Python Coding Practical 2

In this practical we will re-visit how to write linear and non-linear initial value problems in generic form and how to solve them with Python's **odeint** and with the forward and backward Euler methods.

Consider the following example taken from the book by Griffiths and Higham (*Numerical Methods for Ordinary Differential Equations - Initial Value Problems*).

To model a zombie outbreak, denote by

- H(t) the number of humans at time t;
- Z(t) the number of zombies;
- R(t) the number of dead zombies, which can be potentially be resurrected.

We model the following conversion between those three species:

- Humans are turned into zombies at a rate β;
- Zombies are killed by humans at a rate α;
- Dead zombies are resurrected at a rate ζ.

This leads to the following nonlinear differential equation

$$\dot{H}(t) = -\beta\, H(t)Z(t) \qquad \text{(1a)}$$

$$\dot{Z}(t) = \beta\, H(t)Z(t) + \zeta R(t) - \alpha H(t)Z(t) \qquad \text{(1b)}$$

$$\dot{R}(t) = \alpha H(t)Z(t) - \zeta R(t) \qquad \text{(1c)}$$

The initial number of humans, zombies and dead zombies is given by values H_0, Z_0 and R_0 which you will have to make up

Task 1.

Rewrite this system of equations in generic form: define the state-space vector **u**(t), the initial state vector $\mathbf{u_0}$ and the right hand side function **f**(**u**(t),t) such that

$$\dot{\boldsymbol{u}}(t) = \boldsymbol{f}(\boldsymbol{u}(t), t), \boldsymbol{u}(0) = \boldsymbol{u}_0$$

is equivalent to (1).

Task 2.

Solve the initial value problem (1) with **odeint**, forward Euler and backward Euler using **fsolve.** What possible long-term outcomes does the model have when you vary the parameters and initial values?

Using an often employed technique in modelling (often used e.g. when modelling turbulent flows), we can linearise the system (1) using the following assumptions. Split the quantities H(t), Z(t) and R(t) into a constant background value and a time-dependent perturbation, that is

$$H(t) = \bar{H} + \tilde{H}(t)$$

$$Z(t) = \bar{Z} + \tilde{Z}(t)$$

$$R(t) = \bar{R} + \tilde{R}(t)$$

We want to model a small outbreak in a large and previously unaffected human population and set $\bar{Z} = \bar{R} = 0$. Putting this into (1) and ignoring all products of perturbations (which are assumed to be small so that products of them are really small), we get the linearised system

$$\dot{\tilde{H}} = -\beta\bar{H}\tilde{Z}(t) \qquad \text{(2a)}$$

$$\dot{\tilde{Z}} = \beta\bar{H}\tilde{Z}(t) + \zeta\tilde{R}(t) - \alpha\bar{H}\tilde{Z}(t) \qquad \text{(2b)}$$

$$\dot{\tilde{R}} = \alpha\bar{H}\tilde{Z}(t) - \zeta\tilde{R}(t) \qquad \text{(2c)}$$

Task 3.

Rewrite the linearised model (2) in generic form by defining **u** and **f**(**u**(t),t). Then rewrite **f** in matrix form, that is find a matrix M such that

$$\boldsymbol{f}(\boldsymbol{u}(t), t) = M\boldsymbol{u}(t).$$

Task 4.

Solve the linearised model (2) with backward Euler using the backslash operator to solve the system

$$\boldsymbol{u}^{n+1} - \Delta t\, f(\boldsymbol{u}^{n+1}, t_{n+1}) = \boldsymbol{u}^{n}.$$

By varying parameters and initial values, try to develop some insight into when the linear model is a reasonable approximation of the nonlinear model.