**mathematical modelling of Flow-Induced Vibration (FIV)**

**Introduction**

Pipelines conveying fluids play a significant role in modern industry, including chemical processing, power generation and the transportation of commodities such as oil and gas. These are safety critical since system failures can lead to the spillage of fluids which are detrimental to human health and the environment. They are also of enormous economic significance too, so there is clearly a need to identify and address the root causes of failure in pipeline systems. Flow-induced vibration fatigue and failure is one of the most common causes of failure in pipeline systems, accounting for more than 15% of all pipeline failures in Western Europe, Mpofu (2023).

In many practical applications, the operators aim to operate at the highest possible flow velocities to maximise production and profits, however these are associated with more severe vibrations and therefore higher pipe failure rates, Paidoussis (2008). Hence it is very important to be able to identify the safe and optimal operating ranges for a piping system. Studies show that vibration of pipelines results from interactions between a fluid and a structural component, and are thereby influenced by the physical and structural properties of both the fluid and the pipe. The support conditions for the pipes are also found to be very important too. The most stable case is the *clamped-clamped* condition where the ends of the pipes are rigidly attached to physical supports, whereas the least stable support method is the *simply supported-simply* *supported* condition, where the pipe simply lies on top of a support structure but is not rigidly attached to it. The precise form of support method has a large influence on key aspects of pipeline stability, such as the critical flow velocity above which the pipeline becomes unstable and is therefore prone to large FIV and failure.

This document aims to give a brief introduction to mathematical and computational methods that can be used to analyse FIV and covers key areas of model development including model validation and verification.

**Mathematical Modelling of FIV**

A general equation of motion for the vibration of a pipe which is supported for 0≤x≤L is given by

where y(x,t) is the lateral displacement of the pipe at position x and time t, E is the Young’s modulus of the pipe material (Pa), I is the moment of area of the pipe (m4), mp is the mass of the pipe per unit length (kg/m), mf is the mass of the fluid per unit length (kg/m), V is the mean velocity of the fluid flow inside the pipe (m/s).

The first term represents the flexural restoring force. The second term, , centrifugal force due to flow in the curved pipe, the third, , the Coriolis force arising from the relative motion of the pipe and the fluid and the final term, , the inertial force of the pipe and fluid system.

This mathematical model is based on the assumptions that:

1. The pipe behaves like a perfectly elastic beam so that the Young’s modulus is constant
2. The pipe is slender which implies that the amplitude of vibration is small compared to the length
3. The fluid flow is fully developed
4. The fluid is incompressible

Although the fluid is not idealised as inviscid, the equation does not have any term with a viscosity coefficient. It has been shown by Paidoussis (2008) that fluid viscosity does not have a significant effect on the motion.

A further key simplifying assumption is that the Coriolis force term can be neglected in comparison with the other three terms in the equation of motion. This can be justified since many previous studies have neglected the Coriolis term from the equation of motion, e.g. Udoetek (2018) and Yi-min et al (2010), and have found that this leads to an error typically less than 3% in the prediction of natural frequencies that are needed to predict the onset of FIV instabilities.

**This document will consider two important practical cases, those with *simply supported-simply supported* and *clamped-clamped* conditions.**

**FIV for Simply Supported-Simply Supported Conditions**

The boundary conditions are obtained from the nature of the end supports. The case considered here is with a simply supported-simply supported pipe as shown in Figure 1.

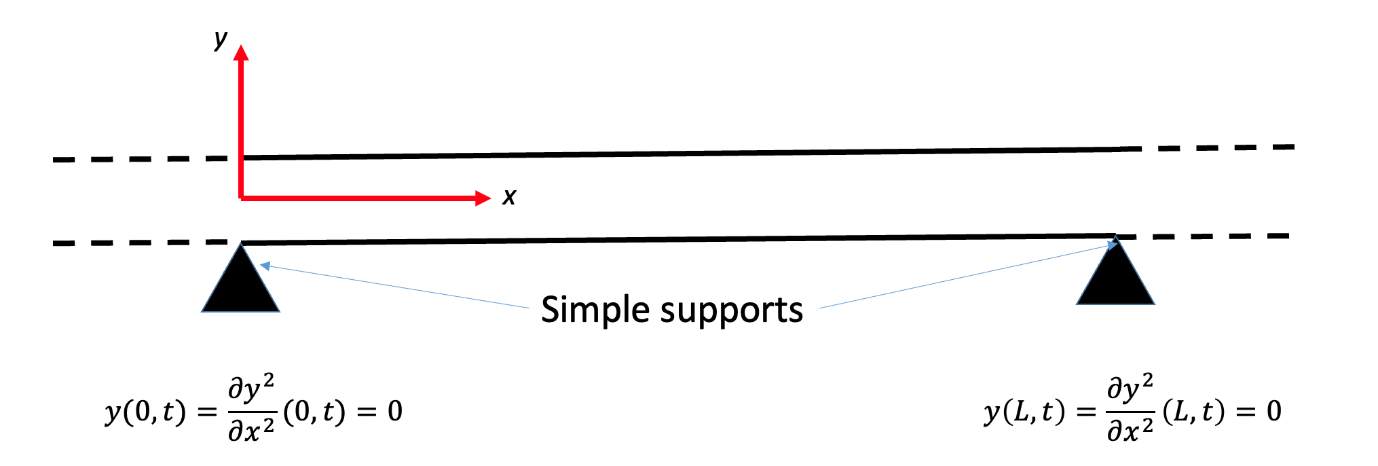


Fig 1: A simply supported-simply supported pipe system.

The boundary conditions to be applied in this case are: at the ends x=0, x=L. Note the second derivative condition indicates that the *bending moment=0* at the simply-supported ends.

**Finite Difference Discretisation**

The finite difference method was used to represent the continuous pipe system as a discrete system. A uniform discretisation was used for the time and spatial domains, as shown in Figure 2. Finite Difference approximations were applied at these discrete nodes.

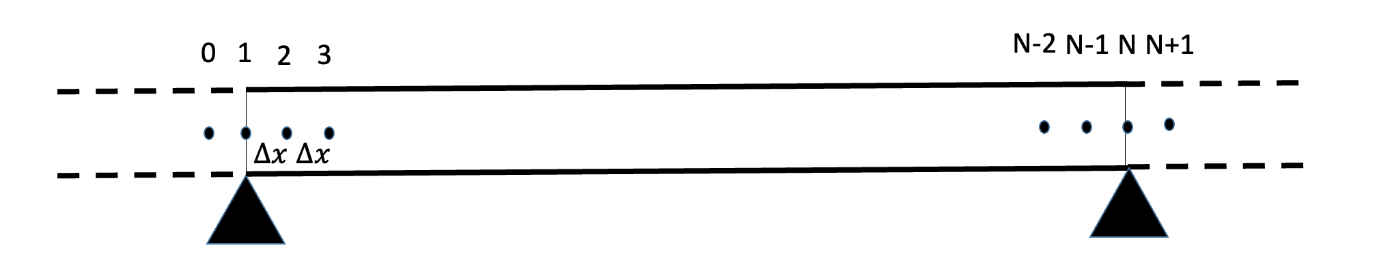


Fig 2: Uniform spatial discretisation of a simply-supported pipe system.

Second order discretisations for the second and fourth order derivative terms are used:

where represents the value of y at the ith node x=xi and Δx is the grid spacing. The same second order expression is used for the second order time derivative.

**Computation of the Natural Frequency**

Since the equation of motion varies in both space and time, the method of separation of variables can be used to obtain an equation in space from which the natural frequency was then computed using the finite difference method. This approach has been used successfully in similar beam models, Rao (2011). The solution is written as

The equation of motion then becomes

This can be re-written as

Hence

where is the total mass of the pipe and fluid per unit length.

Applying a uniform spatial discretisation leads to:

Let and and simplifying the equation leads to:

Putting , , , , this equation can be rewritten as

This was then applied to the internal nodes i=2 to i=N-1, giving the following linear equations:

For i=2

For i=3

For i=4 to i= N-3:

For i=N-2

For i=N-1

We can now simplify these using the boundary conditions, which are: . The second condition is that the bending moment is zero at both simply-supported ends, leading to the second derivative boundary condition that at the clamped ends leads to:

The equations and boundary conditions can then be expressed in the following matrix form:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| C3-C1-λ | C4 | C5 | 0 | … | … | 0 | Y2 |  | 0 |
| C2 | C3-λ | C4 | C5 | 0 | … | … | Y3 |  | 0 |
| C1 | C2 | C3-λ | C4 | C5 | 0 | … | Y4 |  | 0 |
| 0 | … | … | … | … | … | … | Y5 |  | 0 |
| … | … | … | … | … | … | … | … | = | 0 |
| … | … | … | … | … | … | … | … |  | 0 |
| … | 0 | C1 | C2 | C3-λ | C4 | C5 | YN-3 |  | 0 |
| … | … | 0 | C1 | C2 | C3-λ | C4 | YN-2 |  | 0 |
| 0 | … | … | 0 | C1 | C2 | C3-C5-λ | YN-1 |  | 0 |

This is a sparse pentadiagonal matrix of size (N-2)x(N-2) which can be expressed in the form M – where I is the identify matrix and M is an (N-2)x(N-2) matrix given by:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| C3-C1 | C4 | C5 | 0 | … | … | 0 |
| C2 | C3 | C4 | C5 | 0 | … | … |
| C1 | C2 | C3 | C4 | C5 | 0 | … |
| 0 | … | … | … | … | … | … |
| … | … | … | … | … | … | … |
| … | … | … | … | … | … | … |
| … | 0 | C1 | C2 | C3 | C4 | C5 |
| … | … | 0 | C1 | C2 | C3 | C4 |
| 0 | … | … | 0 | C1 | C2 | C3-C5 |

It can be observed that the constant is the eigenvalue which is equal to , where is the natural frequency. The natural frequencies are found by solving the eigenvalue problem so that

**NOTE: The critical velocity at which the pipe loses stability is computed by the condition that one of natural frequencies becomes zero.**

A Python program was developed to determine the 1st and 2nd natural frequencies.

**Verification and Validation of the Finite Difference Solver**

The effect of grid density on the computed natural frequency for an experimental case due to Dodds & Runyan (1965), also reported in Dangal & Ghimire (2019), of an aluminium pipe with L=3.048m, E=68.9 GPa. I=8.73x10-9 (kgm2), mf=0.38 kg/m, mtot=0.715 kg/m, V=13.10 m/s. The grid convergence study in Figure 3 showed the effect of the number of nodes on the calculated natural frequency. This is obtained by running the Python program **gridconvergence\_FIV\_natural\_frequency\_fdm.py** in the **simply-simply** directory.

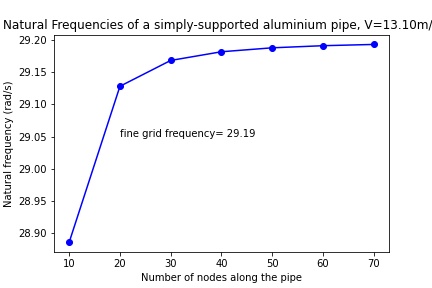


Figure 3: Effect of grid density on the natural frequency of a simply supported-simply supported Aluminium pipe considered by Dodds & Runyan (1965).

It can be seen that the solutions have converged for Nx>50.

The computed fine grid natural frequency of 29.19 rad/s compares with the experimental value of Dodds & Runyan (1965) of 26.10 rad/s: an error of 11.8%. The other experimental cases considered by Dodds & Runyan (1965) are for:

* V=23.485 m/s, the predicted value of 25.25 rad/s compares with the experimental value of 24.11 rad/s – a 4.7% error
* V=29.722 m/s, the predicted value of 21.22 rad/s compares with the average experimental value of 19.93 rad/s – a 6.5% error.

These can be summarised in the following table, which also includes the numerical predictions of Dangal & Ghimire (2019) who used a Finite Element method. Table 1 shows that the numerical method is generally reasonably accurate and compares well with previous relevant studies.

|  |  |  |  |
| --- | --- | --- | --- |
| Flow velocity m/s | Experiment (Dodds & Runyan (1965) rad/s | Finite Element (Dangal & Ghimire (2019) rad/s | Finite Difference model rad/s |
| 13.10 | 26.10 | 29.00 | 29.19 |
| 23.485 | 24.11 | 24.73 | 25.25 |
| 29.722 | 19.93 | 20.47 | 21.11 |

Table 1: Validation of effect of flow velocity on natural frequency for a simply supported-simply supported pipe.

**Order of Accuracy of the Numerical Method**

The order of accuracy of the numerical method can be estimated as follows. If and refer to the natural frequencies calculated using three different grid levels with the same node spacing reduction rate, r, such that the respective grid spacings are , then if is the actual value of the natural frequency then

Hence, taking logarithms of both sides, gives

Comparing the errors using 40, 80 and 160 nodes for the coarse, medium and fine grids, gives an estimate of p=2.04 => indicating it is second order accurate, as expected since these are based on second order finite difference approximations.

**Critical Velocities**

As the velocity of the fluid in the pipe increases, the natural frequency decreases. The *critical velocity for instability* is the velocity for which the natural frequency first becomes equal to zero. The analysis for predicting the natural frequency described above can then be used to predict the critical velocity.

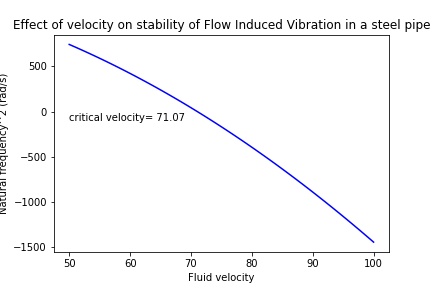


Figure 4: Effect of flow velocity on the natural frequency of a simply supported-simply supported steel pipe considered by Dangal & Ghimire (2019).

The following table compares the predictions of critical velocities for the simply-supported simply-supported case, using the present Finite Difference model with Nx=60 nodes (implemented in the Python program **criticalvelocity\_FIV\_fdm.py** on the **simply-simply** directory) against the Finite Element predictions of Dangal & Ghimire (2019).

|  |  |  |
| --- | --- | --- |
| Critical velocity m/s | Finite Element (Dangal & Ghimire (2019) | Finite Difference model |
| Steel pipe | 70.72 | 71.07 |
| Aluminium pipe | 41.25 | 41.00 |
| CPVC pipe | 8.46 | 8.41 |

Table 2: Comparison of the predictions of critical velocities for simply supported-simply supported steel, aluminium and CPVC pipes, with Nx=60 against the FE predictions of Dangal & Ghimire (2019).

Again, the agreement between the two methods is very good.

**FIV FOR Clamped-Clamped Conditions**

The boundary conditions are obtained from the nature of the end supports. The case considered here is with fixed ends as shown in Figure 5.

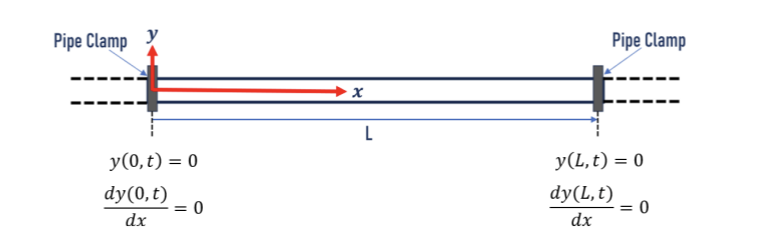


Fig 5: Pipe with clamped ends showing the boundary conditions at the clamps, Mpofu (2023).

**Finite Difference Discretisation**

The finite difference method was used to represent the continuous pipe system as a discrete system. Once again a uniform discretisation was used for the time and spatial domains, as shown in Figure 6. Finite Difference approximations were applied at these discrete nodes as described in the simply supported-simply supported case described above.

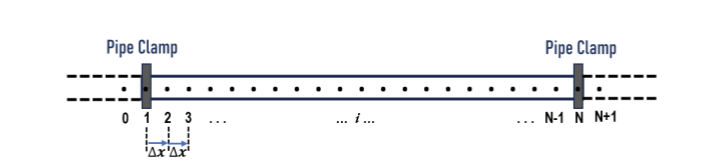


Fig 6: Uniform spatial discretisation of the clamped pipe system, Mpofu (2023).

**Computation of the Natural Frequency**

The analysis proceeds as for the simply supported-simply supported case but with different boundary conditions. Using the boundary conditions, which are: . The derivative boundary condition that at the clamped ends leads to:

The equations and boundary conditions can then be expressed in the following matrix form:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| C1+C3-λ | C4 | C5 | 0 | … | … | 0 | Y2 |  | 0 |
| C2 | C3-λ | C4 | C5 | 0 | … | … | Y3 |  | 0 |
| C1 | C2 | C3-λ | C4 | C5 | 0 | … | Y4 |  | 0 |
| 0 | … | … | … | … | … | … | Y5 |  | 0 |
| … | … | … | … | … | … | … | … | = | 0 |
| … | … | … | … | … | … | … | … |  | 0 |
| … | 0 | C1 | C2 | C3-λ | C4 | C5 | YN-3 |  | 0 |
| … | … | 0 | C1 | C2 | C3-λ | C4 | YN-2 |  | 0 |
| 0 | … | … | 0 | C1 | C2 | C3+C5-λ | YN-1 |  | 0 |

This is a sparse pentadiagonal matrix of size (N-2)x(N-2) which can be expressed in the form M – where I is the identify matrix and M is an (N-2)x(N-2) matrix given by:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| C1+C3 | C4 | C5 | 0 | … | … | 0 |
| C2 | C3 | C4 | C5 | 0 | … | … |
| C1 | C2 | C3 | C4 | C5 | 0 | … |
| 0 | … | … | … | … | … | … |
| … | … | … | … | … | … | … |
| … | … | … | … | … | … | … |
| … | 0 | C1 | C2 | C3 | C4 | C5 |
| … | … | 0 | C1 | C2 | C3 | C4 |
| 0 | … | … | 0 | C1 | C2 | C3+C5 |

As noted above, the constant is the eigenvalue which is equal to , where is the natural frequency. The natural frequencies are found by solving the eigenvalue problem so that

**Once again, the critical velocity at which the pipe loses stability is computed by the condition that one of natural frequencies becomes zero.**

**Verification and Validation of the Finite Difference Solver**

**Natural Frequencies**

For the specific example of a steel pipe from Dangal & Ghimire (2019), use L=3.048m, E=207 GPa. I=8.73x10-9 (m4), mtot=1.386 kg/m, mf=0.38kg/m and V=50m/s. Figure 7 shows a grid convergence study showed the effect of the number of nodes on the natural frequency. This is obtained by running the Python program **gridconvergence\_FIV\_natural\_frequency\_fdm.py** in **the clamped-clamped** directory.

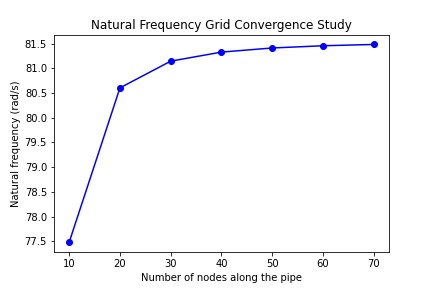


Figure 7: Effect of grid density on the natural frequency of a clamped-clamped steel pipe considered by Dodds & Runyan (1965).

It can be seen that the solutions have converged for Nx>50.

**Order of Accuracy of the Numerical Method**

The order of accuracy of the method can be estimated as follows. If and refer to the natural frequencies calculated using three different grid levels with the same node spacing reduction rate, r, such that the respective grid spacings are , then if is the actual value of the natural frequency then

Hence, taking logarithms of both sides, gives

If we compare the errors using 40, 80 and 160 nodes for the coarse, medium and fine grids, gives an estimate of p=2.039 => indicating second order accurate, as expected since these are based on second order finite difference approximations.

**Prediction of Critical Velocities**

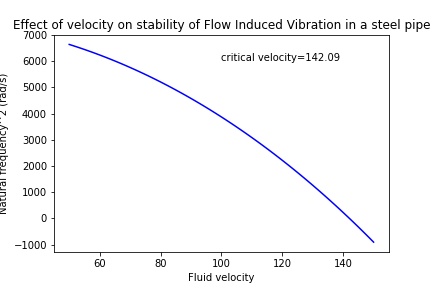


Figure 8: Effect of flow velocity on the natural frequency of a clamped-clamped steel pipe considered by Dangal & Ghimire (2019).

The following table compares the predictions of critical velocities for the clamped-clamped case, using the present Finite Difference model with Nx=60 nodes (implemented in the Python code **criticalvelocity\_FIV\_fdm.py** in the **clamped-clamped** directory) against the Finite Element predictions of Dangal & Ghimire (2019).

|  |  |  |
| --- | --- | --- |
| Critical velocity m/s | Finite Element (Dangal & Ghimire (2019) | Finite Difference model |
| Steel pipe | 141.43 | 142.09 |
| Aluminium pipe | 81.60 | 81.98 |
| CPVC pipe | 16.74 | 16.82 |

Table 2: Comparison of the predictions of critical velocities for clamped-clamped steel, aluminium and CPVC pipes, with Nx=60 against the FE predictions of Dangal & Ghimire (2019).

Again, the agreement between the two methods is very good.

**Calculation of Pipe Displacement and Amplitude of Vibration**

We will use uniform discretisations for the time and spatial domains. In addition to the spatial discretisation we will define a series of times, tj for j=1,…,Nt, where t1=0 and tj=(j-1)Δt in terms of a uniform time-step, Δt. The time response of the displacement of the pipe from the equilibrium position was obtained by applying central finite difference approximations in time and space to the equation

This is first rearranged to

And applying central finite difference approximations yields:

where is the finite difference approximation of the displacement of the pipe at spatial node I and time grid node j. This can be further simplified to obtain the explicit time stepping equation for the solution as follows

where and .

The time-stepping algorithm requires initial values to start the computation. Note that the boundary conditions and lead to the conditions: , and for all j. For example the condition leads to

One of the main challenges of integrating a second order time derivative is to provide initial conditions to enable the time integration to proceed. Here we will use an analytical result from Udoetek (2018) to provide these initial conditions.

**At the second time position j=2 (t=Δt):**

For node i=2:

As described below we can approximate the initial velocity of the pipe by . We can then use a second order discretisation in time to write this as 3

To approximate we use the clamped condition . Using a second order discretisation in space leads to

which leads to

For nodes 2<i<Nx-1:

For node i=Nx-1:

Similar analyses gives us

and

which leads to

**At other times j≥3, t>Δt:**

For node i=2:

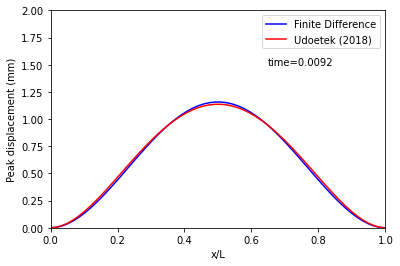
For nodes 2<i<Nx-1:

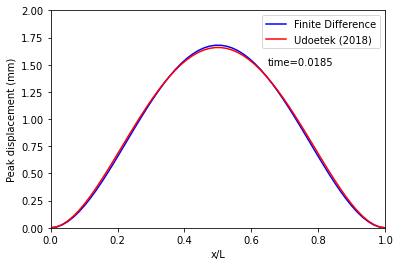
For node i=Nx-1:

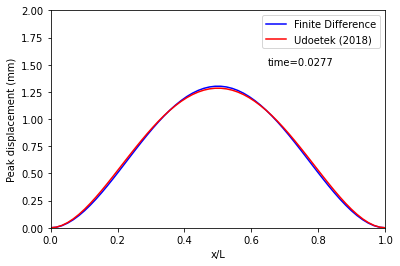
The function used to provide the initial condition on at t=0 is provided by the analytical model of Udoetek (2018). For clamped-clamped conditions this predicted that the displacement of the pipe is given by:

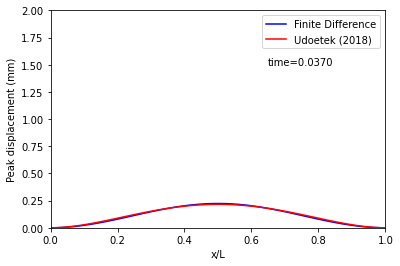
where w is the natural frequency (rad/s) and the peak velocity, u, is measured by a vibration sensor. This leads to the condition:

A python program (**displacement\_FIV\_fdm.py** on the **clamped-clamped** directory) is written to solve the time-dependent equation for y(x,t) subject to the initial conditions,. The following figures show a comparison between the Finite Difference solution and the Udoetek (2018) solution at various times for the case of a clamped-clamped steel pipe with L=3.048m, E=207 GPa. I=8.73x10-9 (m4), mtot=1.386 kg/m, mf=0.38 kg/m and V=50m/s. This uses Nx=60 nodes with Δx=0.0517m, and solves for a whole period 0≤t≤0.0771s with w=81.46 rad/s, Nt=4174 time positions and Δt=0.0000185s.

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**The solutions show the characteristic sinusoidal displacement and the agreement between the Finite Difference and analytical solution of Udoetek (2018) is very good in all cases.**

**Stability of the Finite Difference Method**

Since the Finite Difference solution of the displacement is based on an explicit time-stepping method, this means that the stability of the numerical solution will depend on the relative sizes of the spatial grid spacing and time step. Mpofu (2023) found that the Finite Difference solution is stable provided

This formula gives useful guidance on choosing the relative size of Δx and Δt.

**References**

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