Introduction to Design Optimization

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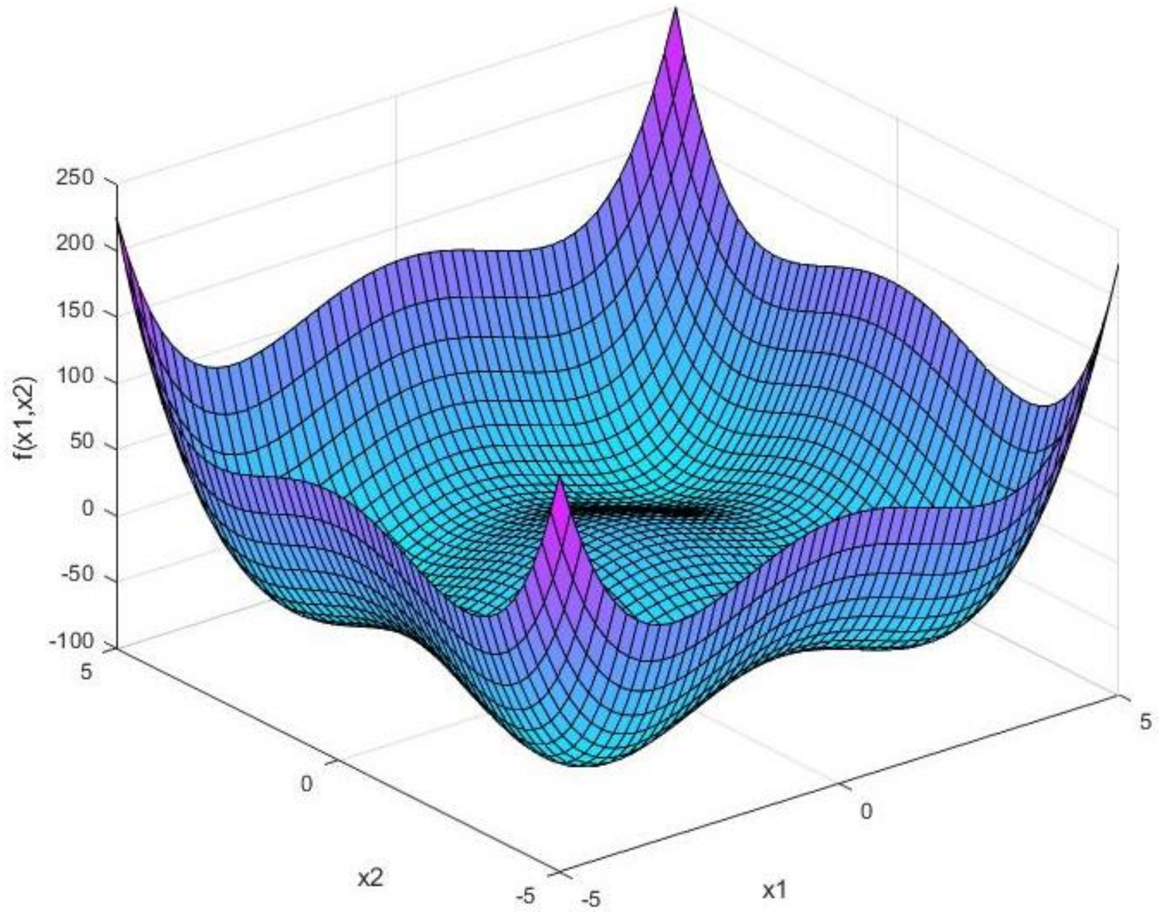
Workshop Problems and Solutions



Styblinski-Tang Function

$$f\left(x\right)=\frac{1}{2}\sum_{i=1}^{d}\left(x_i^4-16_i^2+5x_i\right)$$

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Workshop Problems

1) Questions to be solved using the Graphical method

Minimize: $f\left(x_1,x_2\right)=\left(x_1-3\right)^2+\left(x_2-3\right)^2$ (1)

Subject to: $g_1\left(x_1,x_2\right)=x_1+x_2\leq 4$

$x_1,x_2\geq 0$

Optimal Solution $x_1^*=2, x_2^*=2, f^*(2,2)=2$

Maximize: $f\left(x_1,x_2\right)=x_1+2x_2$ (2)

Subject to: $g_1\left(x_1,x_2\right)=2x_1+x_2\leq 4$

$x_1,x_2\geq 0$

Optimal Solution $x_1^*=0, x_2^*=4, f^*(0,4)=8$

Minimize: $f\left(x_1,x_2\right)=x_1+3x_2$ (3)

Subject to: $g_1\left(x_1,x_2\right)=x_1+4x_2\geq 48$

$g_2\left(x_1,x_2\right)=5x_1+x_2\geq 50$

$x_1,x_2\geq 0$

Optimal Solution $x_1^*=8, x_2^*=10, f^*(8,10)=38$

Maximize: $f\left(x_1,x_2\right)=4x_1x_2$

Subject to: $g_1=x_1+x_2\leq 20$ (4)

$g_2=x_2-x_1\leq 10$

where: $x_1,x_2\geq 0$

Optimal Solution $x_1^*=10, x_2^*=10, f^*(10,10)=400$

Minimize: $f\left(x_1,x_2\right)=5x_1+10x_2$ (5)

Subject to: $g_1\left(x_1,x_2\right)=10x_1+5x_2\leq 50$

$g_2\left(x_1,x_2\right)=5x_1-5x_2\geq -20$

$x_1,x_2\geq 0$

Optimal Solution $x_1^*=0, x_2^*=0, f^*(0,0)=0$

Minimize: $f\left(x_1,x_2\right)=x_1^2-2x_2^2-4x_1$

Subject to: $g_1=x_1+x_2\leq 6$ (6)

$g_2=x_2\leq 3$

where: $x_1,x_2\geq 0$

Optimal Solution $x_1^*=2, x_2^*=3, f^*(2,3)=-22$

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Minimize:	$f(x_1,x_2)=x_1x_2$	
Subject to:	$g_1=x_1+x_2^2\le 0$	(7)
	$g_2=x_1^2+x_2^2\le 9$	
Optimal Solution	$x_1^*=-2.5414, x_2^*=1.5942, f^*(-2.5414,1.5942)=-4.0514$	
Minimize:	$f(x_1,x_2)=3x_1+6x_2$	
Subject to:	$g_1(x_1,x_2)=-3x_1+3x_2\le 2$	(8)
	$g_2(x_1,x_2)=4x_1+2x_2\le 4$	
	$g_3(x_1,x_2)=-x_1+3x_2\ge 1$	
Optimal Solution	$x_1^*=-0.5, x_2^*=0.167, f^*(-0.5,0.167)=-0.5$	
Minimize & Maximize:	$f(x,y)=2x^2+y^2-2xy-3x-2y$	
Subject to:	$g_1(x,y)=y-x\le 0$	(9)
	$h_1(x,y)=x^2+y^2-1=0$	
Optimal Solution	$x_{\min}^*=0.7071, y_{\min}^*=0.7071, f_{\min}^*(0.7071,0.7071)=-3.036$ $x_{\max}^*=-0.7071, y_{\max}^*=-0.7071, f_{\max}^*(0.7071,0.7071)=4.036$	
Minimize & Maximize:	$f(x,y)=4x^2+3y^2-5xy-8x$	(10)
Subject to:	$h_1(x,y)=x+y=4$	
Optimal Solution	$x_{\min}^*=2.167, y_{\min}^*=1.833, f_{\min}^*(2.167,1.833)=-8.33$ No local maximum	
Minimize & Maximize:	$f(x_1,x_2)=9x_1^2+18x_1x_2+13x_2^2-4$	(11)
Subject to:	$h=x_1^2+x_2^2+2x_1=16$	
Optimal Solution	$x_{1_{\min_{Global}}}^*=2.59, x_{2_{\min_{Global}}}^*=-2.02, f_{\min_{Global}}^*(2.59,-2.02)=15.29$ $x_{1_{\min_{Local}}}^*=-3.73, x_{2_{\min_{Local}}}^*=3.09, f_{\min_{Local}}^*(-3.73,3.09)=37.88$ $x_{1_{\max_{Global}}}^*=-3.63, x_{2_{\max_{Global}}}^*=-3.18, f_{\max_{Global}}^*(-3.63,-3.18)=453.2$ $x_{1_{\max_{Local}}}^*=1.51, x_{2_{\max_{Local}}}^*=3.27, f_{\max_{Local}}^*(1.51,3.27)=244.5$	
Minimize & Maximize:	$f(x,y)=2x+3y-x^3-2y^2$	
Subject to:	$g_1(x,y)=x+3y\le 6$	(12)
	$g_2(x,y)=5x+2y\le 10$	
where:	$x,y\ge 0$	

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Optimal Solution

$$x^*_{\min_{Global}} = 2, y^*_{\min_{Global}} = 0, f^*_{\min_{Global}}(2,0) = -4$$
$$x^*_{\min_{Local}} = 0, y^*_{\min_{Local}} = 0, f^*_{\min_{Local}}(0,0) = 0$$
$$x^*_{\min_{Local}} = 0, y^*_{\min_{Local}} = 2, f^*_{\min_{Local}}(0,2) = -2$$
$$x^*_{\min_{Local}} = 1.39, y^*_{\min_{Local}} = 1.54, f^*_{\min_{Local}}(1.39,1.54) = 0$$
$$x^*_{\max_{Global}} = 0.82, y^*_{\max_{Global}} = 0.75, f^*_{\max_{Global}}(0.82,0.75) = 2.21$$

Minimize & Maximize: $f(x_1,x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$ (13)

Subject to: $g_1(x_1,x_2) = x_1 + x_2 \le 3$

Optimal Solution

$$x^*_{1_{\min}} = 2, x^*_{2_{\min}} = 1, f^*_{\min_{Local}}(2,1) = -25$$
$$x^*_{1_{\max}} = -2.31, x^*_{2_{\max}} = 0.33, f^*_{\max_{Local}}(-2.31,0.33) = 24.97$$

No global maximum or minimum

Minimize & Maximize: $f(x,y) = 9x^2 + 13y^2 + 18xy - 4$ (14)

Subject to: $g_1(x,y) = x^2 + y^2 + 2x \ge 16$

Optimal Solution

$$x^*_{\min_{Global}} = 2.59, y^*_{\min_{Global}} = -2.01, f^*_{\min_{Global}}(2.59,-2.01) = 15.25$$
$$x^*_{\min_{Local}} = -3.73, y^*_{\min_{Local}} = 3.09, f^*_{\min_{Local}}(-3.73,3.09) = 37.87$$

No Local Maximum

2) Lecture Example Problems to be solved using the Graphical method

Example 01: Design of a Can

Minimize: $S(D,H) = \pi DH + \frac{\pi}{2}D^2$ (15)

Subject to: $g_1(D,H) = 400 - \frac{\pi}{4}D^2H \le 0$

such that: $3.5 \le D \le 8$

$8 \le H \le 18$

Optimal Solution $D^*_{\min} = 7.98, H^*_{\min} = 8, S^*_{\min_{Global}}(7.98,8) = 300.53$

Example 02: Design of a Rectangular Beam

Minimize: $A(b,d) = bd$ (16)

Subject to: $g_1(b,d) = 840 \times 10^6 - 165bd^2 \le 0$

$g_2(b,d) = 36,000 - 50bd \le 0$

$g_3(b,d) = d - 2b \le 0$

such that: $b,d \ge 0$

Optimal Solution $b^*_{\min} = 108.371, d^*_{\min} = 216.741, S^*_{\min_{Global}}(108.371,216.741) = 23488.41$

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Solutions to Workshop Problems

1) Solution to Graphical Problems

Solution to Problem 1

$$\begin{aligned} \textbf{Minimize:} \quad & f(x_1,x_2) = (x_1-3)^2 + (x_2-3)^2 \\ \textbf{Subject to:} \quad & g_1(x_1,x_2) = x_1 + x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned} \qquad (1)$$

Optimal Solution $\quad x_1^* = 2, x_2^* = 2, f^*(2,2) = 2$

Rearrange equations in order to plot x_2 vs x_1.

Inequality Constraint:

$$g_1(x_1,x_2) = x_1 + x_2 \le 4 \quad \Rightarrow \quad x_2 \le 4 - x_1$$

Limits

$$x_1, x_2 \ge 0$$

Objective Function:

$$f(x_1,x_2) = (x_1-3)^2 + (x_2-3)^2$$

$$\Rightarrow (x_2-3)^2 = f - (x_1-3)^2$$

$$x_2 - 3 = \pm\sqrt{f - (x_1-3)^2}$$

$$\therefore x_2 = 3 \pm \sqrt{f - (x_1-3)^2}$$

Calculating the inequality constraint between the limits of 0 and 6 for x_1.

	g1	fmin	2
x1	**x2 g1**	**x2+ fmin**	**x2- fmin**
0	4	#NUM!	#NUM!
0.1	3.9	#NUM!	#NUM!
⋮	⋮	⋮	⋮
1.9	2.1	3.89	2.11
2	**2**	**4.00**	**2.00**
2.1	1.9	4.09	1.91
2.2	1.8	4.17	1.83
⋮	⋮	⋮	⋮
3.9	0.1	4.09	1.91
4	0	4.00	2.00
⋮	⋮	⋮	⋮
5.8	-1.8	#NUM!	#NUM!
5.9	-1.9	#NUM!	#NUM!
6	-2	#NUM!	#NUM!

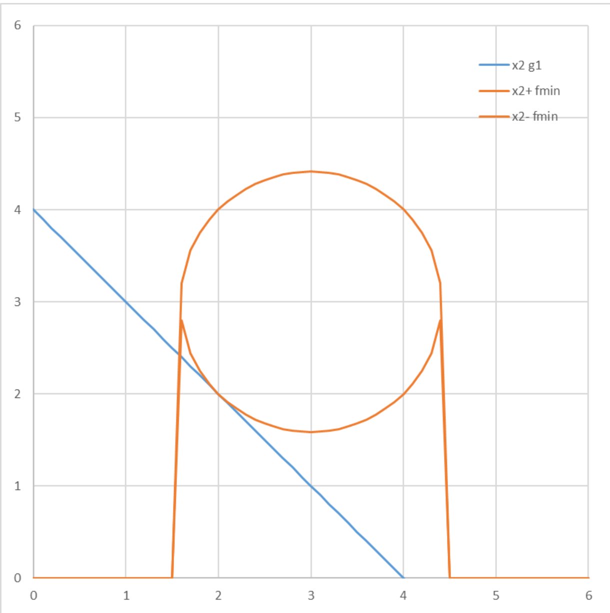

Figure Example 1: Plot of solution to problem 1, with only a single contour of the objective function at the optimal point (2,2)

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Solution to Problem 2

$$\textbf{Maximize:} \quad f(x_1, x_2) = x_1 + 2x_2 \qquad (2)$$

$$\textbf{Subject to:} \quad g_1(x_1, x_2) = 2x_1 + x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Optimal Solution $\quad x_1^* = 0, x_2^* = 4, f^*(0,4) = 8$

Rearrange equations in order to plot x_2 vs x_1.

Inequality Constraint:

$$g_1(x_1, x_2) = 2x_1 + x_2 \le 4 \quad \Rightarrow \quad x_2 \le 4 - 2x_1$$

Limits

$$x_1 \ge 0 \qquad x_2 \ge 0$$

Objective Function:

$$f(x_1, x_2) = x_1 + 2x_2$$

$$\Rightarrow f - x_1 = 2x_2$$

$$2x_2 = f - x_1$$

$$\therefore x_2 = \frac{f - x_1}{2}$$

Calculating the inequality constraint between the limits of 0 and 2 for x_1.

		fmin
	g1	8
x1	x2 g1	x2 fmin
0.0	4.0	4.00
0.1	3.8	3.95
0.2	3.6	3.90
0.3	3.4	3.85
0.4	3.2	3.80
0.5	3.0	3.75
0.6	2.8	3.70
0.7	2.6	3.65
0.8	2.4	3.60
0.9	2.2	3.55
1.0	2.0	3.50
1.1	1.8	3.45
1.2	1.6	3.40
1.3	1.4	3.35
1.4	1.2	3.30
1.5	1.0	3.25
1.6	0.8	3.20
1.7	0.6	3.15
1.8	0.4	3.10
1.9	0.2	3.05
2.0	0.0	3.00

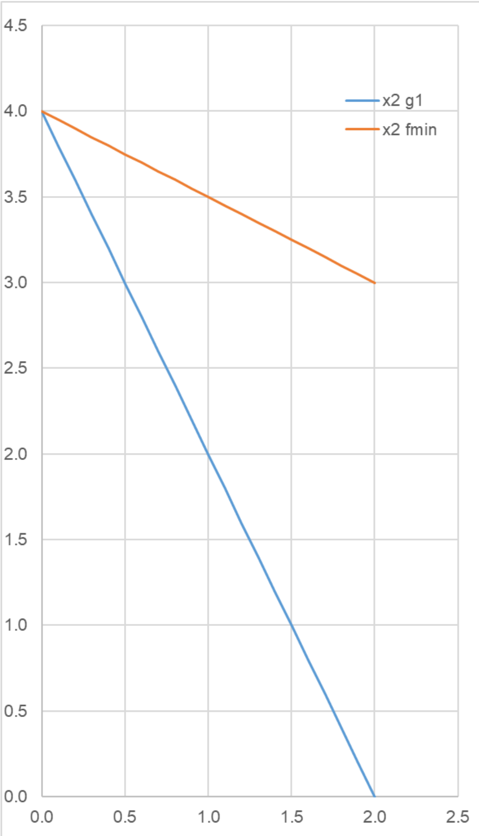

Figure Example 2: Plot of solution to problem 2, with only a single contour of the objective function at the optimal point (0,4)

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Solution to Problem 3

$$\begin{aligned} \textbf{Minimize:} \quad & f(x_1,x_2) = x_1 + 3x_2 \\ \textbf{Subject to:} \quad & g_1(x_1,x_2) = x_1 + 4x_2 \geq 48 \\ & g_2(x_1,x_2) = 5x_1 + x_2 \geq 50 \\ & x_1, x_2 \geq 0 \end{aligned} \qquad (3)$$

Optimal Solution $\quad x_1^* = 8, x_2^* = 10, f^*(8,10) = 38$

Rearrange equations in order to plot x_2 vs x_1.

Inequality Constraints:

$$g_1(x_1,x_2) = x_1 + 4x_2 \geq 48 \quad \Rightarrow \quad 4x_2 \geq 48 - x_1$$

$$\therefore x_2 \geq \frac{48 - x_1}{4}$$

$$g_2(x_1,x_2) = 5x_1 + x_2 \geq 50$$

$$\Rightarrow x_2 \geq 50 - 5x_1$$

Limits

$$x_1, x_2 \geq 0$$

Objective Function:

$$f(x_1,x_2) = x_1 + 3x_2$$

$$\Rightarrow f - x_1 = 3x_2$$

$$\Rightarrow 3x_2 = f - x_1$$

$$\therefore x_2 = \frac{f - x_1}{3}$$

Calculating the inequality constraint between the limits of 0 and 50 for x_1.

			fmin
	g1	**g2**	**38**
x1	**x2 g1**	**x2 g2**	**x2 fmin**
0.0	12.0	50.0	12.7
2.0	11.5	40.0	12.0
4.0	11.0	30.0	11.3
6.0	10.5	20.0	10.7
8.0	**10.0**	**10.0**	**10.0**
10.0	9.5	0.0	9.3
12.0	9.0	-10.0	8.7
⋮	⋮	⋮	⋮
48.0	0.0	-190.0	-3.3
50.0	-0.5	-200.0	-4.0

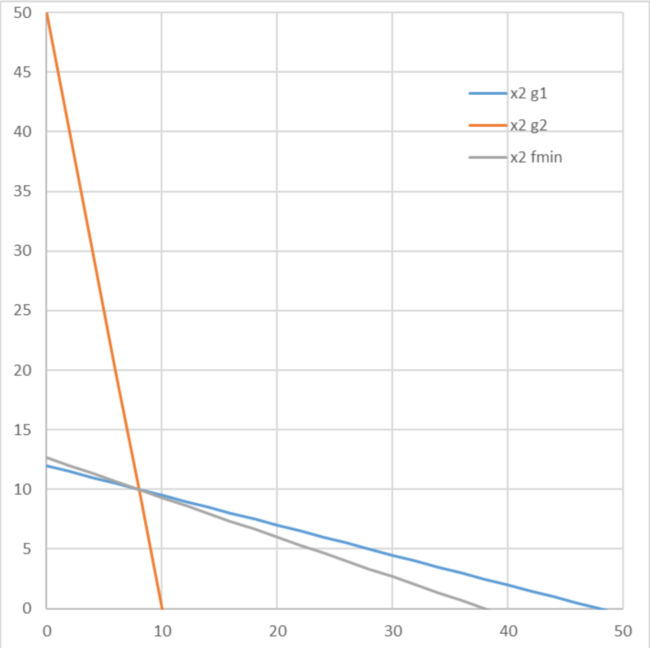

Figure Example 3: Plot of solution to problem 3, with only a single contour of the objective function at the optimal point (8,10)

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Solution to Problem 4

$$\textbf{Maximize:} \quad f(x_1,x_2) = 4x_1x_2$$

$$\textbf{Subject to:} \quad g_1 = x_1 + x_2 \le 20 \qquad (4)$$

$$g_2 = x_2 - x_1 \le 10$$

$$\textbf{where:} \quad x_1, x_2 \ge 0$$

Optimal Solution $x_1^* = 10, x_2^* = 10, f^*(10,10) = 400$

Rearrange equations in order to plot x_2 vs x_1.

Inequality Constraints:

$$g_1 = x_1 + x_2 \le 20$$
$$\therefore x_2 \le 20 - x_1$$
$$g_2 = x_2 - x_1 \le 10$$
$$\therefore x_2 \le 10 + x_1$$

Limits

$$x_1, x_2 \ge 0$$

Objective Function:

$$f(x_1,x_2) = 4x_1x_2$$
$$\Rightarrow \frac{f}{4x_1} = x_2$$
$$\therefore x_2 = \frac{f}{4x_1}$$

Calculating the inequality constraint between the limits of 0 and 20 for x_1.

		f =	100	200	300	400	500
x_1	x_2 (g_1)	x_2 (g_2)	x_2 (f = 100)	x_2 (f = 200)	x_2 (f = 300)	x_2 (f = 400)	x_2 (f = 500)
0.0	20.0	10.0	2.50E+20	5.00E+20	7.50E+20	1.00E+21	1.25E+21
1.0	19.0	11.0	25.0	50.0	75.0	100.0	125.0
2.0	18.0	12.0	12.5	25.0	37.5	50.0	62.5
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
8.0	12.0	18.0	3.1	6.3	9.4	12.5	15.6
9.0	11.0	19.0	2.8	5.6	8.3	11.1	13.9
10.0	**10.0**	20.0	2.5	5.0	7.5	**10.0**	12.5
11.0	9.0	21.0	2.3	4.5	6.8	9.1	11.4
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
19.0	1.0	29.0	1.3	2.6	3.9	5.3	6.6
20.0	0.0	30.0	1.3	2.5	3.8	5.0	6.3

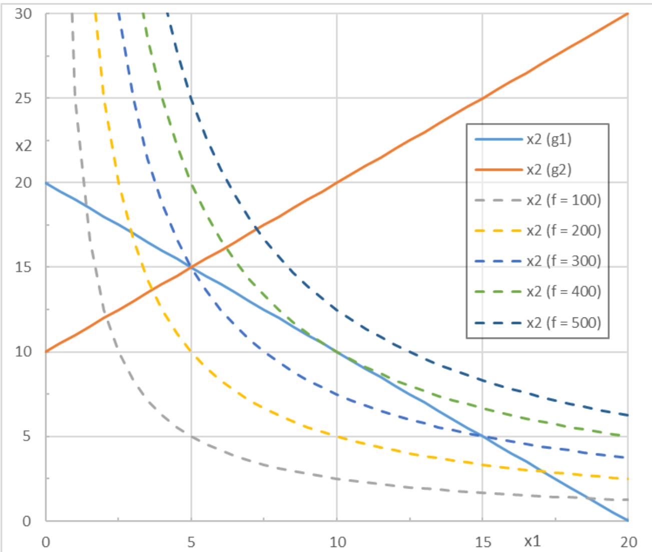

Figure Example 4: Plot of solution to problem 4, with five contour of the objective function showing the optimum at (10,10)

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Solution to Problem 5

$$\textbf{Minimize: } f(x_1,x_2) = 5x_1 + 10x_2 \tag{5}$$

$$\textbf{Subject to: } g_1(x_1,x_2) = 10x_1 + 5x_2 \le 50$$

$$g_2(x_1,x_2) = 5x_1 - 5x_2 \ge -20$$

$$x_1, x_2 \ge 0$$

Optimal Solution $x_1^* = 0, x_2^* = 0, f^*(0,0) = 0$

Rearrange equations in order to plot x_2 vs x_1.

Inequality Constraints:

$$g_1(x_1,x_2) = 10x_1 + 5x_2 \le 50, \quad \Rightarrow 5x_2 \le 50 - 10x_1 \Rightarrow x_2 \le \frac{50 - 10x_1}{5}$$

$$\therefore x_2 \le 10 - 2x_1$$

$$g_2(x_1,x_2) = 5x_1 - 5x_2 \ge -20$$

$$\Rightarrow 5x_1 + 20 \ge 5x_2$$

$$\Rightarrow 5x_2 \le 5x_1 + 20 \Rightarrow x_2 \le \frac{5x_1 + 20}{5}$$

$$\therefore x_2 \le x_1 + 4$$

Limits

$$x_1 \ge 0 \qquad x_2 \ge 0$$

Objective Function:

$$f(x_1,x_2) = 5x_1 + 10x_2$$

$$\Rightarrow f - 5x_1 = 10x_2 \Rightarrow 10x_2 = f - 5x_1$$

$$\therefore x_2 = \frac{f - 5x_1}{10}$$

Calculating the inequality constraint between the limits of 0 and 3 for x_1.

		f =	0	25	50	75	100
x_1	**x_2 (g_1)**	**x_2 (g_2)**	**x_2 (f = 0)**	**x_2 (f = 25)**	**x_2 (f = 50)**	**x_2 (f =75)**	**x_2 (f = 100)**
0.0	10.0	4.0	**0.0**	2.5	5.0	7.5	10.0
0.2	9.6	4.2	-0.1	2.4	4.9	7.4	9.9
0.4	9.2	4.4	-0.2	2.3	4.8	7.3	9.8
0.6	8.8	4.6	-0.3	2.2	4.7	7.2	9.7
0.8	8.4	4.8	-0.4	2.1	4.6	7.1	9.6
1.0	8.0	5.0	-0.5	2.0	4.5	7.0	9.5
1.2	7.6	5.2	-0.6	1.9	4.4	6.9	9.4
1.4	7.2	5.4	-0.7	1.8	4.3	6.8	9.3
1.6	6.8	5.6	-0.8	1.7	4.2	6.7	9.2
1.8	6.4	5.8	-0.9	1.6	4.1	6.6	9.1
2.0	6.0	6.0	-1.0	1.5	4.0	6.5	9.0
2.2	5.6	6.2	-1.1	1.4	3.9	6.4	8.9
2.4	5.2	6.4	-1.2	1.3	3.8	6.3	8.8
2.6	4.8	6.6	-1.3	1.2	3.7	6.2	8.7
2.8	4.4	6.8	-1.4	1.1	3.6	6.1	8.6
3.0	4.0	7.0	-1.5	1.0	3.5	6.0	8.5

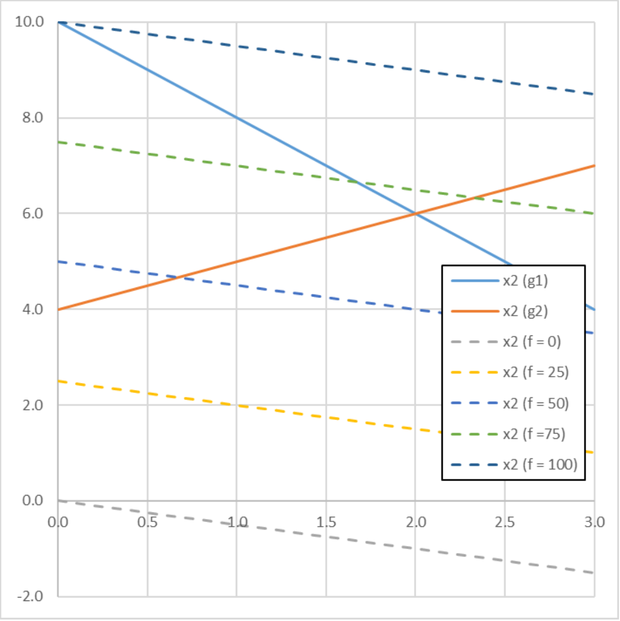

Figure Example 5: Plot of solution to problem 5, with 5 contour of the objective function showing optimum at (0,0)

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Solution to Problem 6

Minimize: $f(x_1, x_2) = x_1^2 - 2x_2^2 - 4x_1$

Subject to: $g_1 = x_1 + x_2 \le 6$ (6)

$g_2 = x_2 \le 3$

where: $x_1, x_2 \ge 0$

Optimal Solution $x_1^* = 2, x_2^* = 3, f^*(2,3) = -22$

Inequality Constraints:

$$g_1 = x_1 + x_2 \le 6$$
$$\therefore x_2 \le 6 - x_1$$

Limits

$$x_1 \ge 0 \quad 0 \le x_2 \le 3$$

Objective Function:

$$f(x_1, x_2) = x_1^2 - 2x_2^2 - 4x_1$$
$$\Rightarrow f = x_1^2 - 2x_2^2 - 4x_1$$
$$\Rightarrow 2x_2^2 = x_1^2 - 4x_1 - f$$
$$\Rightarrow x_2^2 = \frac{x_1^2 - 4x_1 - f}{2}$$
$$\therefore x_2 = \pm\sqrt{\frac{x_1^2 - 4x_1 - f}{2}}$$

Calculating the inequality constraint between the limits of 0 and 6 for x_1.

		f =	-22	-10	-5	-4
x1	**x2(g1)**	**x2(g2)**	**x2+ (f=-22)**	**x2+ (f=-10)**	**x2+ (f=-5)**	**x2+ (f=-4)**
0	6.00	3.0	3.32	2.24	1.58	1.41
0.25	5.75	3.0	3.25	2.13	1.43	1.24
0.50	5.50	3.0	3.18	2.03	1.27	1.06
0.75	5.25	3.0	3.13	1.94	1.13	0.88
1.00	5.00	3.0	3.08	1.87	1.00	0.71
1.25	4.75	3.0	3.05	1.81	0.88	0.53
1.50	4.50	3.0	3.02	1.77	0.79	0.35
1.75	4.25	3.0	3.01	1.74	0.73	0.18
2.00	4.00	3.0	3.00	1.73	0.71	0.00
2.25	3.75	3.0	3.01	1.74	0.73	0.18
2.50	3.50	3.0	3.02	1.77	0.79	0.35
2.75	3.25	3.0	3.05	1.81	0.88	0.53
3.00	3.00	3.0	3.08	1.87	1.00	0.71
3.25	2.75	3.0	3.13	1.94	1.13	0.88
3.50	2.50	3.0	3.18	2.03	1.27	1.06
3.75	2.25	3.0	3.25	2.13	1.43	1.24
4.00	2.00	3.0	3.32	2.24	1.58	1.41
4.25	1.75	3.0	3.40	2.35	1.74	1.59
4.50	1.50	3.0	3.48	2.47	1.90	1.77
4.75	1.25	3.0	3.58	2.60	2.07	1.94
5.00	1.00	3.0	3.67	2.74	2.24	2.12
5.25	0.75	3.0	3.78	2.88	2.40	2.30
5.50	0.50	3.0	3.89	3.02	2.57	2.47
5.75	0.25	3.0	4.00	3.17	2.74	2.65
6.00	0.00	3.0	4.12	3.32	2.92	2.83

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Plot of optimal solution

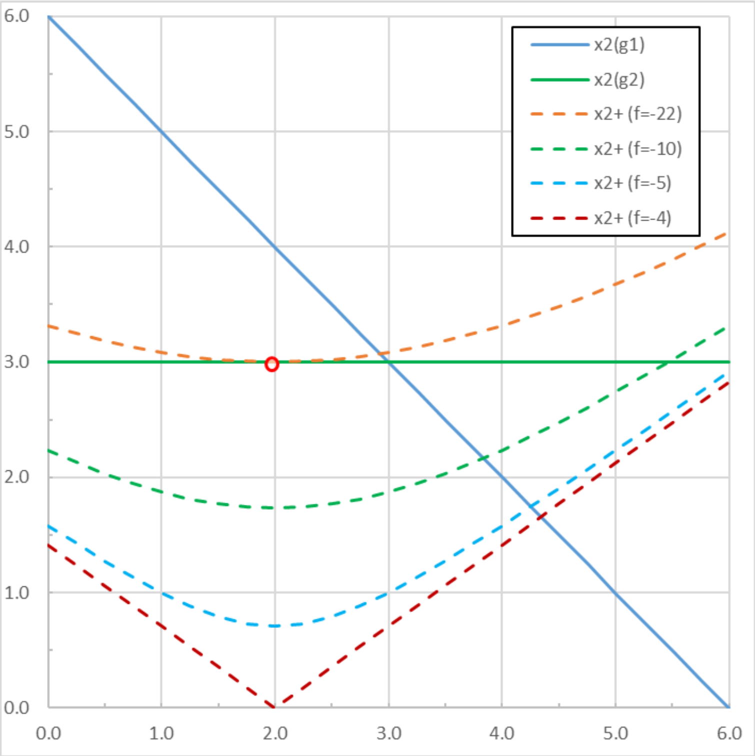

Figure Example 6: Plot of solution to problem 6, with four contour of the objective function showing the optimum at (2,3)

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Solution to Problem 7

$$\textbf{Minimize:} \quad f(x_1,x_2) = x_1 x_2$$

$$\textbf{Subject to:} \quad g_1 = x_1 + x_2^2 \le 0 \qquad (7)$$

$$g_2 = x_1^2 + x_2^2 \le 9$$

Optimal Solution $x_1^* = -2.5414, x_2^* = 1.5942, f^*(-2.5414, 1.5942) = -4.0514$

Inequality Constraints:

$$g_1 = x_1 + x_2^2 \le 0, \quad \Rightarrow x_2^2 \le -x_1, \quad \therefore x_2 \le \pm\sqrt{-x_1}$$

$$g_2 = x_1^2 + x_2^2 \le 9, \quad \Rightarrow x_2^2 \le 9 - x_1^2, \quad \therefore x_2 \le \pm\sqrt{9 - x_1^2}$$

Objective Function:

$$f(x_1,x_2) = x_1 x_2, \quad \Rightarrow \frac{f}{x_1} = x_2, \quad \therefore x_2 = \frac{f}{x_1}$$

Calculating the inequality constraint between the limits of 0 and 6 for x_1.

					fmin	fmax
					-4.0514	4.0514
x1	**x2(+g1)**	**x2(-g1)**	**x2(+g2)**	**x2(-g2)**	**x2 - fmin**	**x2 - fmax**
0.0	0.003	-0.003	3.000	-3.000	405139	-405139.0
-0.10	0.316	-0.316	2.998	-2.998	40.510	-40.510
-0.20	0.447	-0.447	2.993	-2.993	20.256	-20.256
-0.30	0.548	-0.548	2.985	-2.985	13.504	-13.504
-0.40	0.632	-0.632	2.973	-2.973	10.128	-10.128
-0.50	0.707	-0.707	2.958	-2.958	8.103	-8.103
-0.60	0.775	-0.775	2.939	-2.939	6.752	-6.752
-0.70	0.837	-0.837	2.917	-2.917	5.788	-5.788
-0.80	0.894	-0.894	2.891	-2.891	5.064	-5.064
-0.90	0.949	-0.949	2.862	-2.862	4.501	-4.501
-1.00	1.000	-1.000	2.828	-2.828	4.051	-4.051
-1.10	1.049	-1.049	2.791	-2.791	3.683	-3.683
-1.20	1.095	-1.095	2.750	-2.750	3.376	-3.376
-1.30	1.140	-1.140	2.704	-2.704	3.116	-3.116
-1.40	1.183	-1.183	2.653	-2.653	2.894	-2.894
-1.50	1.225	-1.225	2.598	-2.598	2.701	-2.701
-1.60	1.265	-1.265	2.538	-2.538	2.532	-2.532
-1.70	1.304	-1.304	2.472	-2.472	2.383	-2.383
-1.80	1.342	-1.342	2.400	-2.400	2.251	-2.251
-1.90	1.378	-1.378	2.322	-2.322	2.132	-2.132
-2.00	1.414	-1.414	2.236	-2.236	2.026	-2.026
-2.10	1.449	-1.449	2.142	-2.142	1.929	-1.929
-2.20	1.483	-1.483	2.040	-2.040	1.842	-1.842
-2.30	1.517	-1.517	1.926	-1.926	1.761	-1.761
-2.40	1.549	-1.549	1.800	-1.800	1.688	-1.688
-2.54	**1.594**	**-1.594**	**1.594**	-1.594	1.594	**-1.594**
-2.60	1.612	-1.612	1.497	-1.497	1.558	-1.558
-2.70	1.643	-1.643	1.308	-1.308	1.501	-1.501
-2.80	1.673	-1.673	1.077	-1.077	1.447	-1.447
-2.90	1.703	-1.703	0.768	-0.768	1.397	-1.397
-3.00	1.732	-1.732	0.008	-0.008	1.350	-1.350

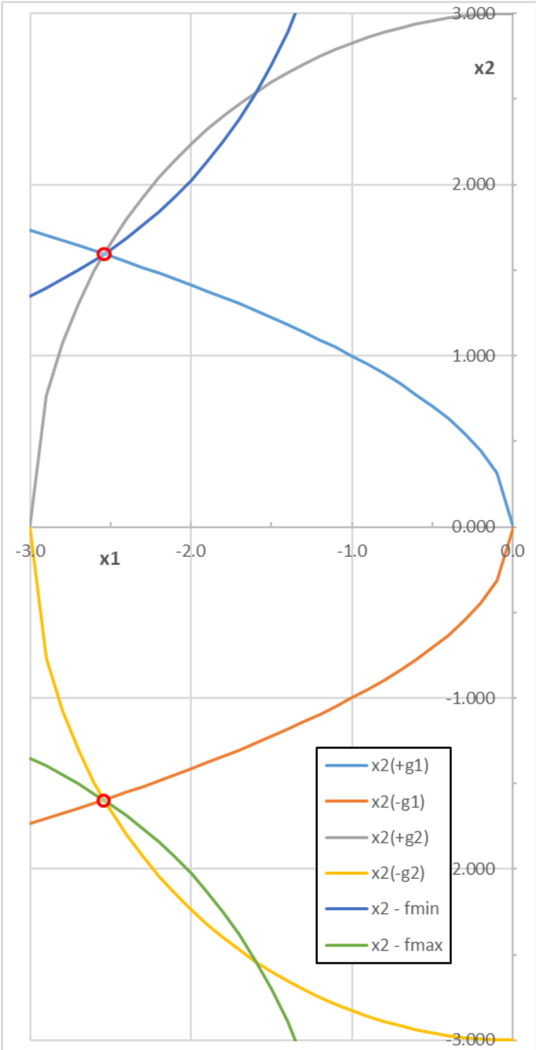

Figure Example 7: Plot of solution to problem 7, with only the contour of the optimal value of the objective function showing the optimum maximum and minimum values at (-2.54,±1.594)

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Solution to Problem 8

$$\textbf{Minimize: } f(x_1,x_2)=3x_1+6x_2 \tag{8}$$

$$\textbf{Subject to: } g_1(x_1,x_2)=-3x_1+3x_2\le 2$$

$$g_2(x_1,x_2)=4x_1+2x_2\le 4$$

$$g_3(x_1,x_2)=-x_1+3x_2\ge 1$$

Optimal Solution $x_1^*=-0.5, x_2^*=0.167, f^*(-0.5,0.167)=-0.5$

Inequality Constraints:

$$g_1(x_1,x_2)=-3x_1+3x_2\le 2, \quad \Rightarrow 3x_2\le 2+3x_1, \quad \therefore x_2\le \frac{2}{3}+x_1$$

$$g_2(x_1,x_2)=4x_1+2x_2\le 4, \quad \Rightarrow 2x_2\le 4-4x_1, \quad \Rightarrow x_2\le \frac{4-4x_1}{2}, \quad \therefore x_2\le 2-2x_1$$

$$g_3(x_1,x_2)=-x_1+3x_2\ge 1, \quad \Rightarrow 3x_2\ge 1+x_1, \quad \therefore x_2\ge \frac{1+x_1}{3}$$

Objective Function:

$$f(x_1,x_2)=3x_1+6x_2, \quad \Rightarrow f-3x_1=6x_2, \quad \therefore x_2=\frac{f-3x_1}{6}$$

Calculating the inequality constraint between the limits of –1 and 1 for x_1.

				fmin	fmax
	g1	**g2**	**g3**	**-0.5**	**8**
x1	**x2(g1)**	**x2(g2)**	**x2(g3)**	**x2 fmin**	**x2 fmax**
-1	-0.333	4	0.000	0.417	1.833
-0.9	-0.233	3.8	0.033	0.367	1.783
-0.8	-0.133	3.6	0.067	0.317	1.733
-0.7	-0.033	3.4	0.100	0.267	1.683
-0.6	0.067	3.2	0.133	0.217	1.633
-0.5	**0.167**	3	**0.167**	**0.167**	1.583
-0.4	0.267	2.8	0.200	0.117	1.533
-0.3	0.367	2.6	0.233	0.067	1.483
-0.2	0.467	2.4	0.267	0.017	1.433
-0.1	0.567	2.2	0.300	-0.033	1.383
0	0.667	2	0.333	-0.083	1.333
0.1	0.767	1.8	0.367	-0.133	1.283
0.2	0.867	1.6	0.400	-0.183	1.233
0.3	0.967	1.4	0.433	-0.233	1.183
0.444	**1.111**	**1.111**	0.481	-0.306	**1.111**
0.5	1.167	1	0.500	-0.333	1.083
0.6	1.267	0.8	0.533	-0.383	1.033
0.7	1.367	0.6	0.567	-0.433	0.983
0.8	1.467	0.4	0.600	-0.483	0.933
0.9	1.567	0.2	0.633	-0.533	0.883
1	1.667	0	0.667	-0.583	0.833

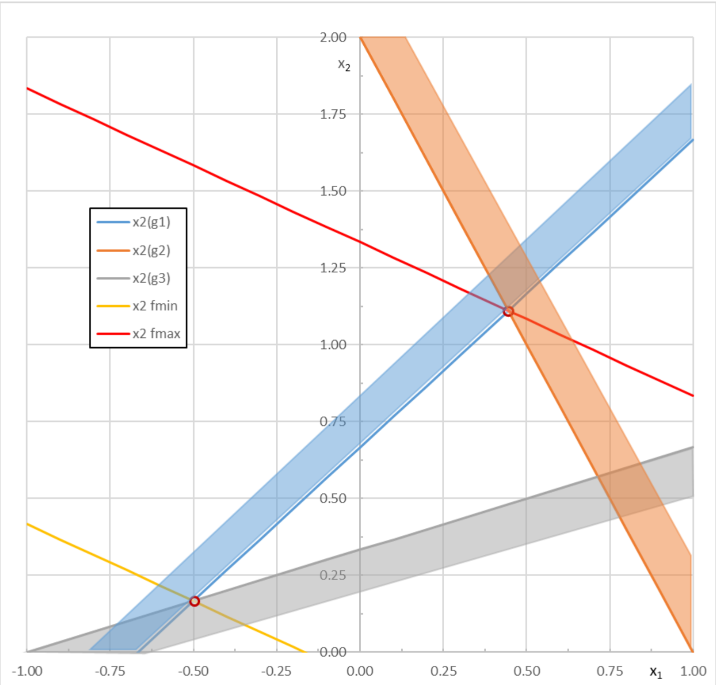

Figure Example 8: Plot of solution to problem 8, with only the contour of the optimal value of the objective function showing the optimum maximum (0.444,1,111) and minimum (-0.5, 0.167) values.

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Solution to Problem 9

$$
\begin{array}{rl}
\textbf{Minimize \& Maximize:} & f(x,y)=2x^2+y^2-2xy-3x-2y \\
\textbf{Subject to:} & g_1(x,y)=y-x\le 0 \\
& h_1(x,y)=x^2+y^2-1=0
\end{array}
\tag{9}
$$

Optimal Solution
$$x^*_{\min}=0.7071, y^*_{\min}=0.7071, f^*_{\min}(0.7071, 0.7071)=-3.036$$
$$x^*_{\max}=-0.7071, y^*_{\max}=-0.7071, f^*_{\max}(0.7071, 0.7071)=4.036$$

Inequality Constraint:

$$g_1(x,y)=y-x\le 0$$
$$\therefore y\le x$$

Equality Constraint:

$$h_1(x,y)=x^2+y^2-1=0$$
$$\Rightarrow y^2=1-x^2$$
$$\therefore y=\pm\sqrt{1-x^2}$$

Objective Function:

$$f(x,y)=2x^2+y^2-2xy-3x-2y$$
$$\Rightarrow 2x^2+y^2-2xy-3x-2y-f=0$$
$$\Rightarrow y^2-2xy-2y+2x^2-3x-f=0$$
$$\therefore y^2-2(x+1)y+\left(2x^2-3x-f\right)=0$$

This rearranged objective function is in the form of a quadratic equation of the form:

$$ay^2+by+c=0$$

where:

$$a=1$$
$$b=-2(x+1)$$
$$c=2x^2-3x-f$$

The solution to this quadratic equation is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Which then gives that the solution of y is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\left(-2(x+1)\right)\pm\sqrt{\left(-2(x+1)\right)^2-4\times 1\times\left(2x^2-3x-f\right)}}{2\times 1}$$
$$\therefore y=(x+1)\pm\sqrt{(x+1)^2-\left(2x^2-3x-f\right)}$$

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Calculating the inequality constraint between the limits of –1 and 1 for x_1.

				fmin		fmax	
	g1	h1		-3.0355		4.0355	
x	y(g1)	y(+h1)	y(-h1)	y(+fmin)	y(-fmin)	y(+fmax)	y(-fmax)
-1.000	-1.000	0.000	0.000	#NUM!	#NUM!	#NUM!	#NUM!
-0.975	-0.975	0.222	-0.222	#NUM!	#NUM!	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-0.725	-0.725	0.689	-0.689	#NUM!	#NUM!	1.216	-0.666
-0.707	**-0.707**	0.707	**-0.707**	#NUM!	#NUM!	1.293	**-0.707**
-0.675	-0.675	0.738	-0.738	#NUM!	#NUM!	1.423	-0.773
-0.650	-0.650	0.760	-0.760	#NUM!	#NUM!	1.517	-0.817
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.650	0.650	0.760	-0.760	2.540	0.760	4.454	-1.154
0.675	0.675	0.738	-0.738	2.615	0.735	4.495	-1.145
0.707	**0.707**	**0.707**	-0.707	2.707	**0.707**	4.548	-1.134
0.725	0.725	0.689	-0.689	2.756	0.694	4.577	-1.127
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.950	0.950	0.312	-0.312	3.296	0.604	4.930	-1.030
0.975	0.975	0.222	-0.222	3.349	0.601	4.968	-1.018
1.000	1.000	#NUM!	#NUM!	3.402	0.598	5.006	-1.006

<u>Plot of optimal solutions</u>

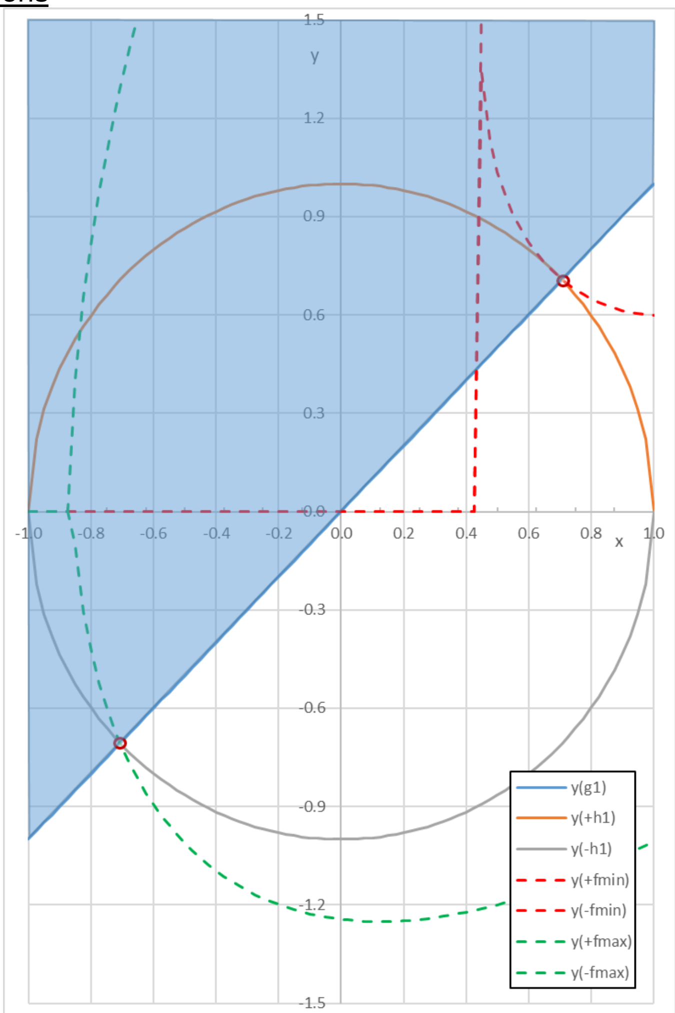

Figure Example 9: Plot of solution to problem 9, with the contour of the objective function at the maximum and minimum optimal values

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Solution to Problem 10

Minimize & Maximize: $f\left(x,y\right)=4x^2+3y^2-5xy-8x$ (10)

Subject to: $h_1\left(x,y\right)=x+y=4$

Optimal Solution $x^*_{\min}=2.167, y^*_{\min}=1.833, f^*_{\min}(2.167,1.833)=-8.33$

No local maximum

<u>Equality Constraint:</u>

$$h_1\left(x,y\right)=x+y=4$$
$$\therefore y=4-x$$

<u>Objective Function:</u>

$$f\left(x,y\right)=4x^2+3y^2-5xy-8x$$
$$\Rightarrow 4x^2+3y^2-5xy-8x-f=0$$
$$\Rightarrow 3y^2-5xy+4x^2-8x-f=0$$
$$\therefore 3y^2-5xy+\left(4x^2-8x-f\right)=0$$

This rearranged objective function is in the form of a quadratic equation of the form:

$$ay^2+by+c=0$$

where:

$$a=3$$
$$b=-5x$$
$$c=4x^2-8x-f$$

The solution to this quadratic equation is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Which then gives that the solution of y is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\left(-5x\right)\pm\sqrt{\left(-5x\right)^2-4\times 3\times\left(4x^2-8x-f\right)}}{2\times 3}$$
$$\therefore y=\frac{5x\pm\sqrt{-23x^2+96x+12f}}{6}$$

Calculating the inequality constraint between the limits of –4 and 4 for x_1.

		fmin		**fmax**		**fmax**		**fmax**		**fmax**	
	h1	**100.00**		**45.00**		**20.00**		**0.00**		**-8.33**	
x	**y(h)**	**y(+f100)**	**y(-f100)**	**y(+f45)**	**y(-f45)**	**y(+f20)**	**y(-f20)**	**y(+f0)**	**y- f0**	**y(+f-8.33)**	**y(-f-8.33)**
-4.00	8.00	0.194	-6.861	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
-3.99	7.99	0.214	-6.864	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
-3.98	7.98	0.233	-6.866	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.05	1.95	7.718	-4.301	5.925	-2.509	4.782	-1.365	3.376	0.040	1.771	1.645
2.10	1.90	7.760	-4.260	5.967	-2.467	4.824	-1.324	3.418	0.082	1.819	1.681
2.167	**1.833**	7.815	-4.204	6.022	-2.411	4.879	-1.268	3.472	0.139	**1.833**	1.778
2.20	1.80	7.842	-4.176	6.049	-2.383	4.906	-1.239	3.499	0.168	#NUM!	#NUM!
2.25	1.75	7.883	-4.133	6.090	-2.340	4.946	-1.196	3.538	0.212	#NUM!	#NUM!
2.30	1.70	7.924	-4.091	6.130	-2.297	4.986	-1.153	3.576	0.257	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
3.90	0.10	9.082	-2.582	7.210	-0.710	5.961	0.539	4.076	2.424	#NUM!	#NUM!
3.95	0.05	9.114	-2.531	7.237	-0.654	5.981	0.602	4.043	2.540	#NUM!	#NUM!
4.00	0.00	9.145	-2.479	7.263	-0.597	6.000	0.667	4.000	2.667	#NUM!	#NUM!

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Plot of optimal solutions

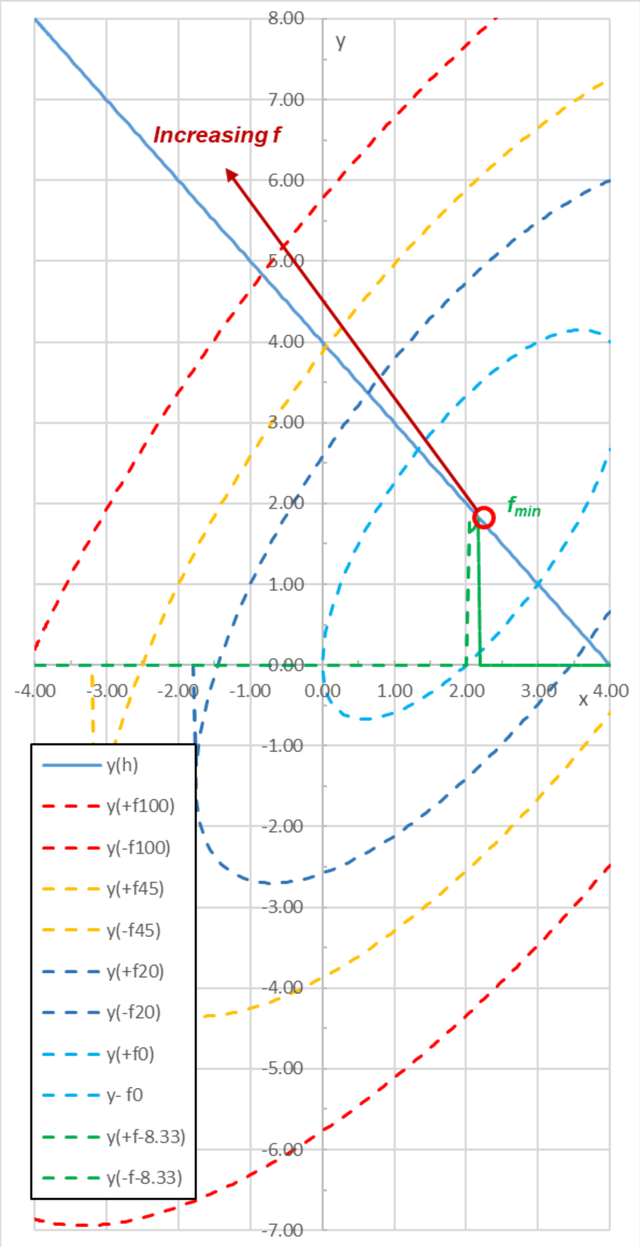

Figure Example 10: Plot of solution to problem 10, with the contours of the objective function showing the minimum optimal values and how there is no local maximum optimum

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Solution to Problem 11

Minimize & Maximize:	$f\left(x_1,x_2\right)=9x_1^2+18x_1x_2+13x_2^2-4$	(11)
Subject to:	$h=x_1^2+x_2^2+2x_1=16$	
Optimal Solution	$x^*_{1_{\min_{Global}}}=2.59, x^*_{2_{\min_{Global}}}=-2.02, f^*_{\min_{Global}}(2.59,-2.02)=15.29$ $x^*_{1_{\min_{Local}}}=-3.73, x^*_{2_{\min_{Local}}}=3.09, f^*_{\min_{Local}}(-3.73,3.09)=37.88$ $x^*_{1_{\max_{Global}}}=-3.63, x^*_{2_{\max_{Global}}}=-3.18, f^*_{\max_{Global}}(-3.63,-3.18)=453.2$ $x^*_{1_{\max_{Local}}}=1.51, x^*_{2_{\max_{Local}}}=3.27, f^*_{\max_{Local}}(1.51,3.27)=244.5$	

Equality Constraint:

$$h=x_1^2+x_2^2+2x_1=16$$
$$\Rightarrow x_2^2=16-x_1^2-2x_1$$
$$\therefore x_2=\pm\sqrt{16-x_1^2-2x_1}$$

Objective Function:

$$f\left(x_1,x_2\right)=9x_1^2+18x_1x_2+13x_2^2-4$$
$$\Rightarrow 9x_1^2+18x_1x_2+13x_2^2-4-f=0$$
$$\Rightarrow 13x_2^2+18x_1x_2+9x_1^2-4-f=0$$
$$\therefore 13x_2^2+\left(18x_1\right)x_2+\left(9x_1^2-4-f\right)=0$$

This rearranged objective function is in the form of a quadratic equation of the form:

$$ay^2+by+c=0$$

where:

$$a=13$$
$$b=18x_1$$
$$c=9x_1^2-4-f$$

The solution to this quadratic equation is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Which then gives that the solution of y is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-18x_1\pm\sqrt{\left(18x_1\right)^2-4\times 13\times\left(9x_1^2-4-f\right)}}{2\times 13}$$
$$\Rightarrow y=\frac{-18x\pm\sqrt{324x_1^2-468x_1^2+208+52f}}{26}$$
$$\therefore y=\frac{-18x\pm\sqrt{-144x_1^2+208+52f}}{26}$$

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Calculating the inequality constraint between the limits of –5.1 and 3.1 for x_1.

	Solution 1		Solution 2		Solution 3		Solution 4
f(x1,x2) =	244.527	**f(x1,x2) =**	37.877	37.88309	15.2909		453.152
x1 =	1.509		-3.732	2	2.5945		-3.630
x2 =	3.272		3.088	-2.924	-2.0198		-3.175

		f =	15.291	**f =**	37.877	**f =**	244.527	**f =**	453.152	
x_1	**x_2(+h)**	**x_2(-h)**	**x_2(+f15.3)**	**x_2(-f15.3)**	**x_2(+f37.9)**	**x_2(-f37.9)**	**x_2(+f244)**	**x_2(-f244)**	**x_2(+f453)**	**x_2(-f453)**
-5.1	0.000	0.000	#NUM!	#NUM!	#NUM!	#NUM!	7.225	-0.131	8.985	-1.891
-5.0	1.000	-1.000	#NUM!	#NUM!	#NUM!	#NUM!	7.175	-0.252	8.924	-2.001
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-3.7	3.116	-3.116	#NUM!	#NUM!	3.114	2.009	6.587	-1.464	8.240	-3.117
-3.63	3.175	**-3.175**	#NUM!	#NUM!	3.157	1.869	6.552	-1.526	8.202	**-3.175**
-3.5	3.279	-3.279	#NUM!	#NUM!	3.205	1.641	6.486	-1.640	8.129	-3.283
-3.4	3.353	-3.353	#NUM!	#NUM!	3.225	1.483	6.435	-1.727	8.073	-3.365
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.4	2.332	-2.332	-1.155	-2.168	-0.249	-3.074	2.568	-5.891	4.164	-7.487
2.5	2.179	-2.179	-1.340	-2.121	-0.356	-3.106	2.487	-5.948	4.086	-7.547
2.595	2.020	**-2.020**	-1.573	**-2.020**	-0.459	-3.133	2.409	-6.001	4.012	-7.604
2.7	1.819	-1.819	#NUM!	#NUM!	-0.578	-3.161	2.322	-6.060	3.928	-7.667
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
3.1	0.436	-0.436	#NUM!	#NUM!	-1.063	-3.230	1.985	-6.278	3.609	-7.901
3.1	0.000	0.000	#NUM!	#NUM!	-1.093	-3.232	1.966	-6.290	3.590	-7.914

<u>Plot of optimal solutions</u>

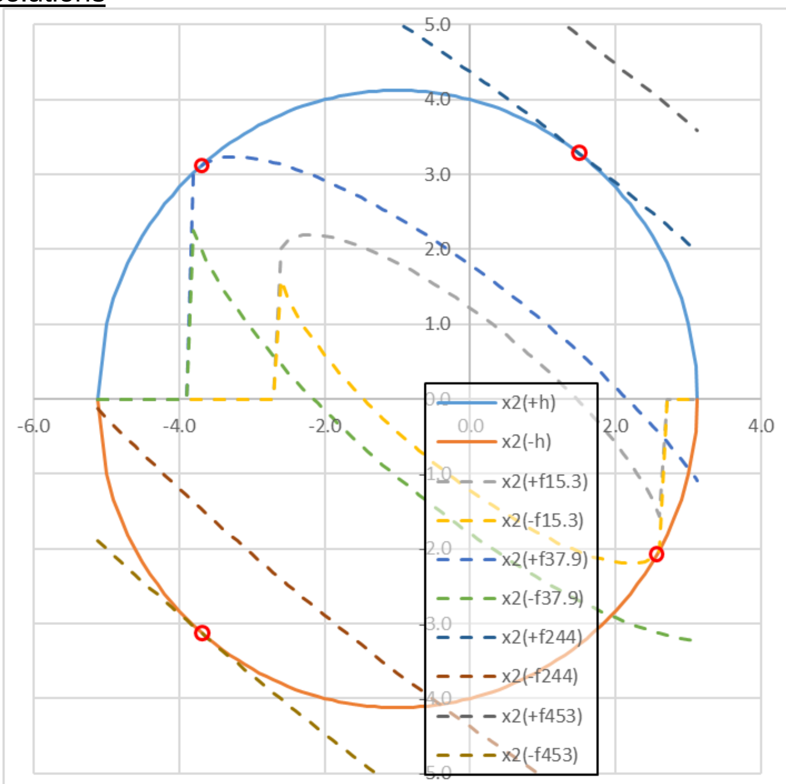

Figure Example 11: Plot of solution to problem 11, with the contours of the objective function showing the global and local maximums and minimum values and their corresponding contours

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Solution to Problem 12

Minimize & Maximize:	$f\left(x,y\right)=2x+3y-x^3-2y^2$	(12)
Subject to:	$g_1\left(x,y\right)=x+3y\le 6$	
	$g_2\left(x,y\right)=5x+2y\le 10$	
where:	$x,y\ge 0$	

Optimal Solution

$$x^*_{\min_{Global}}=2, y^*_{\min_{Global}}=0, f^*_{\min_{Global}}(2,0)=-4$$
$$x^*_{\min_{Local}}=0, y^*_{\min_{Local}}=0, f^*_{\min_{Local}}(0,0)=0$$
$$x^*_{\min_{Local}}=0, y^*_{\min_{Local}}=2, f^*_{\min_{Local}}(0,2)=-2$$
$$x^*_{\min_{Local}}=1.39, y^*_{\min_{Local}}=1.54, f^*_{\min_{Local}}(1.39,1.54)=0$$
$$x^*_{\max_{Global}}=0.82, y^*_{\max_{Global}}=0.75, f^*_{\max_{Global}}(0.82,0.75)=2.21$$

Inequality Constraints:

$$g_1\left(x,y\right)=x+3y\le 6,\quad \Rightarrow 3y\le 6-x,\quad \therefore y\le\frac{6-x}{3}$$
$$g_2\left(x,y\right)=5x+2y\le 10,\quad \Rightarrow 2y\le 10-5x,\quad \therefore y\le\frac{10-5x}{2}$$

Limits

$$x,y\ge 0$$

Objective Function:

$$f\left(x,y\right)=2x+3y-x^3-2y^2$$
$$\Rightarrow f+2y^2+x^3-3y-2x=0$$
$$\Rightarrow 2y^2-3y+x^3-2x+f=0$$
$$\therefore 2y^2-3y+\left(x^3-2x+f\right)=0$$

This rearranged objective function is in the form of a quadratic equation of the form:

$$ay^2+by+c=0$$

where:

$a=2$ $\quad b=-3$ $\quad c=x^3-2x-f$

The solution to this quadratic equation is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Which then gives that the solution of y is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{3\pm\sqrt{\left(-3\right)^2-4\times 2\times\left(x^3-2x+f\right)}}{2\times 2}$$
$$\Rightarrow y=\frac{3\pm\sqrt{9-8x^3+16x-8f}}{4}$$
$$\therefore y=\frac{3\pm\sqrt{-8x^3+16x-8f+9}}{4}$$

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Global Minimum								Global Maximum	
Solution 1		**Solution 2**		**Solution 3**		**Solution 4**		**Solution 5**	
f(x,y) =	-4.000	**f(x,y) =**	0.000	**f(x,y) =**	-2.0000	**f(x,y) =**	-0.0036	**f(x,y) =**	2.2137
g_1(x,y) =	2.000	**g_1(x,y) =**	0.000	**g_1(x,y) =**	6.000	**g_1(x,y) =**	6.000	**g_1(x,y) =**	3.066
g_2(x,y) =	10.000	**g_2(x,y) =**	0.000	**g_2(x,y) =**	4.000	**g_2(x,y) =**	10.000	**g_2(x,y) =**	5.582
x =	2.000	**x =**	0.000	**x =**	0.0000	**x =**	1.3846	**x =**	0.8165
y =	0.000	**y =**	0.000	**y =**	2.0000	**y =**	1.5385	**y =**	0.7500

Calculating the inequality constraint between the limits of 0 and 2 for x_1.

			Solution 1		**Solution 2**		**Solution 3**		**Solution 4**		**Global Maximum**	
		f =	-4.000	**f =**	0.000	**f =**	-2.000	**f =**	-0.004	**f =**	2.214	
x	**y(g_1)**	**y(g_2)**	**y(+f-4)**	**y(-f-4)**	**y(+f0)**	**y(-f0)**	**y(+f-2)**	**y(-f-2)**	**y(+f0)**	**y(-f0)**	**y(+f2.21)**	**y(-f2.21)**
0.0	2.000	5.000	2.351	-0.851	1.500	0.000	2.000	-0.500	1.501	-0.001	#NUM!	#NUM!
0.01	1.997	4.975	2.354	-0.854	1.507	-0.007	2.004	-0.504	1.508	-0.008	#NUM!	#NUM!
0.02	1.993	4.950	2.357	-0.857	1.513	-0.013	2.008	-0.508	1.514	-0.014	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.79	1.737	3.025	2.512	-1.012	1.802	-0.302	2.201	-0.701	1.803	-0.303	#NUM!	#NUM!
0.80	1.733	3.000	2.513	-1.013	1.802	-0.302	2.201	-0.701	1.803	-0.303	#NUM!	#NUM!
0.8165	1.728	2.959	2.513	-1.013	1.802	-0.302	2.201	-0.701	1.803	-0.303	0.757	0.743
0.82	1.727	2.950	2.513	-1.013	1.802	-0.302	2.201	-0.701	1.803	-0.303	0.756	0.744
0.83	1.723	2.925	2.513	-1.013	1.802	-0.302	2.201	-0.701	1.803	-0.303	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.36	1.547	1.600	2.382	-0.882	1.565	-0.065	2.040	-0.540	1.566	-0.066	#NUM!	#NUM!
1.37	1.543	1.575	2.377	-0.877	1.554	-0.054	2.033	-0.533	1.555	-0.055	#NUM!	#NUM!
1.3846	1.538	1.538	2.369	-0.869	1.537	-0.037	2.023	-0.523	1.538	-0.038	#NUM!	#NUM!
1.39	1.537	1.525	2.365	-0.865	1.531	-0.031	2.019	-0.519	1.532	-0.032	#NUM!	#NUM!
1.40	1.533	1.500	2.360	-0.860	1.518	-0.018	2.011	-0.511	1.520	-0.020	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.99	1.337	0.025	1.532	-0.032	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
2.00	1.333	**0.000**	1.500	**0.000**	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!

Plot of optimal solutions

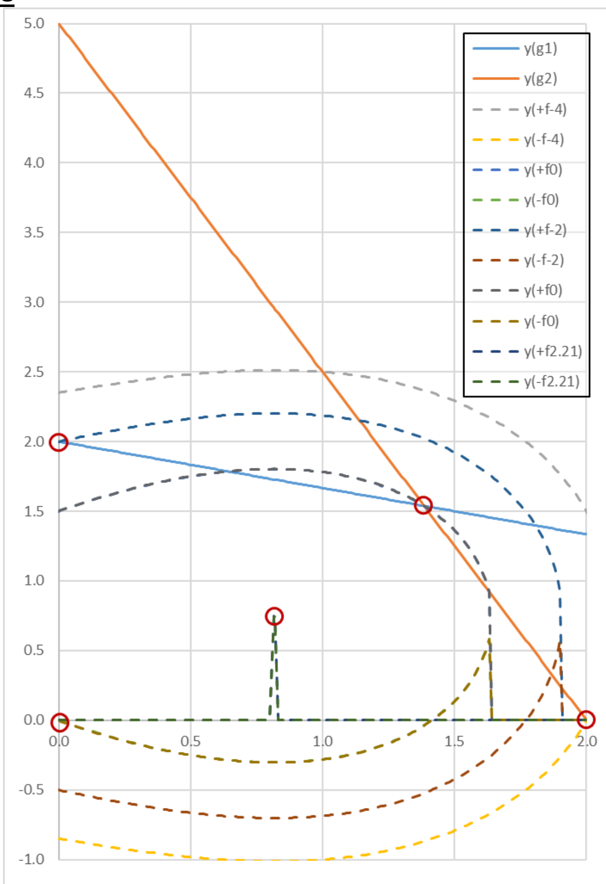

Figure Example 12: Plot of solution to problem 12, with the contours of the objective function showing the global and local maximums and minimum values and their corresponding contours

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Solution to Problem 13

Minimize & Maximize:	$f\left(x_1,x_2\right)=x_1^3-16x_1+2x_2-3x_2^2$	(13)
Subject to:	$g_1\left(x_1,x_2\right)=x_1+x_2\le 3$	
Optimal Solution	$x_{1_{\min}}^*=2, x_{2_{\min}}^*=1, f_{\min_{Local}}^*(2,1)=-25$ $x_{1_{\max}}^*=-2.31, x_{2_{\max}}^*=0.33, f_{\max_{Local}}^*(-2.31, 0.33)=24.97$ No global maximum or minimum	

Inequality Constraints:

$g_1\left(x_1,x_2\right)=x_1+x_2\le 3, \quad \therefore x_2\le 3-x_1$

Objective Function:

$f\left(x_1,x_2\right)=x_1^3-16x_1+2x_2-3x_2^2$

$\Rightarrow f-x_1^3+16x_1-2x_2+3x_2^2=0$

$\Rightarrow 3x_2^2-2x_2-x_1^3+16x_1+f=0$

$\therefore 3x_2^2-2x_2-\left(x_1^3-16x_1-f\right)=0$

This rearranged objective function is in the form of a quadratic equation of the form:

$ay^2+by+c=0$

where:

$a=3 \qquad b=-2 \qquad c=-\left(x_1^3-16x_1-f\right)$

The solution to this quadratic equation is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Which then gives that the solution of y is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{(-2)^2-4\times 3\times\left\{-\left(x_1^3-16x_1-f\right)\right\}}}{2\times 3}$$

$$\Rightarrow y=\frac{2\pm\sqrt{4+4\times 3\times\left(x_1^3-16x_1-f\right)}}{6}$$

$$\Rightarrow y=\frac{2\pm 2\sqrt{1+3\times\left(x_1^3-16x_1-f\right)}}{6} \qquad \therefore y=\frac{1\pm 1\sqrt{1+3x_1^3-48x_1-3f}}{3}$$

Solution for Maximum		Solution for Minimum: There are two global minimum!!			
x1 =	-2.3094	**x1 =**	2	and also	-1
x2 =	0.333334	**x2 =**	1		4
f(x) =	24.967	**f(x) =**	-25		-25
g1 =	-1.97607	**g1 =**	3		3

Calculating constraints and objective function values for x_2 between limits of -3 and 3 for x_1

		fmin		f		fmax	
	g1	-25.0		20.0		24.966	
x1	x2(g1)	x2(+f-25)	x2(-f-25)	x2(+f-20)	x2(-f20)	x2(+f24.9)	x2(-f24.9)
-3.0	6.0	-3.597	4.263	-0.333	1.000	#NUM!	#NUM!
-2.9	5.9	-3.639	4.306	-0.551	1.217	#NUM!	#NUM!

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⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-2.4	5.4	-3.745	4.412	-0.946	1.613	#NUM!	#NUM!
-2.3	5.3	-3.748	4.414	-0.953	1.620	**0.323**	**0.344**
-2.2	5.2	-3.744	4.411	-0.943	1.609	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-1.2	4.2	-3.444	4.111	#NUM!	#NUM!	#NUM!	#NUM!
-1.1	4.1	-3.391	4.057	#NUM!	#NUM!	#NUM!	#NUM!
-1.0	**4.0**	-3.333	**4.000**	#NUM!	#NUM!	#NUM!	#NUM!
-0.9	3.9	-3.272	3.939	#NUM!	#NUM!	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.8	1.2	-0.555	1.221	#NUM!	#NUM!	#NUM!	#NUM!
1.9	1.1	-0.440	1.106	#NUM!	#NUM!	#NUM!	#NUM!
2.0	**1.0**	-0.333	**1.000**	#NUM!	#NUM!	#NUM!	#NUM!
2.1	0.9	-0.242	0.909	#NUM!	#NUM!	#NUM!	#NUM!
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.9	0.1	-0.719	1.386	#NUM!	#NUM!	#NUM!	#NUM!
3.0	0.0	-0.869	1.535	#NUM!	#NUM!	#NUM!	#NUM!

Plot of optimal solutions

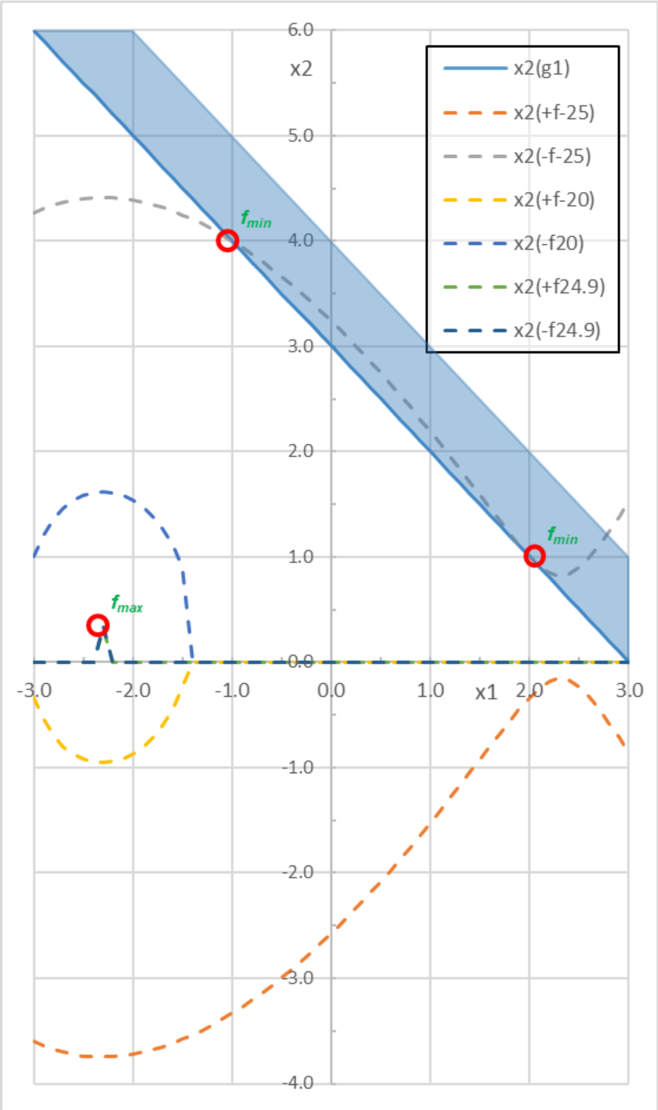

Figure Example 13: Plot of solution to problem 13, with the contours of the objective function showing the local maximum and minimum values, their corresponding contours and an extra contour for f=20

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3D Plot of the objective function

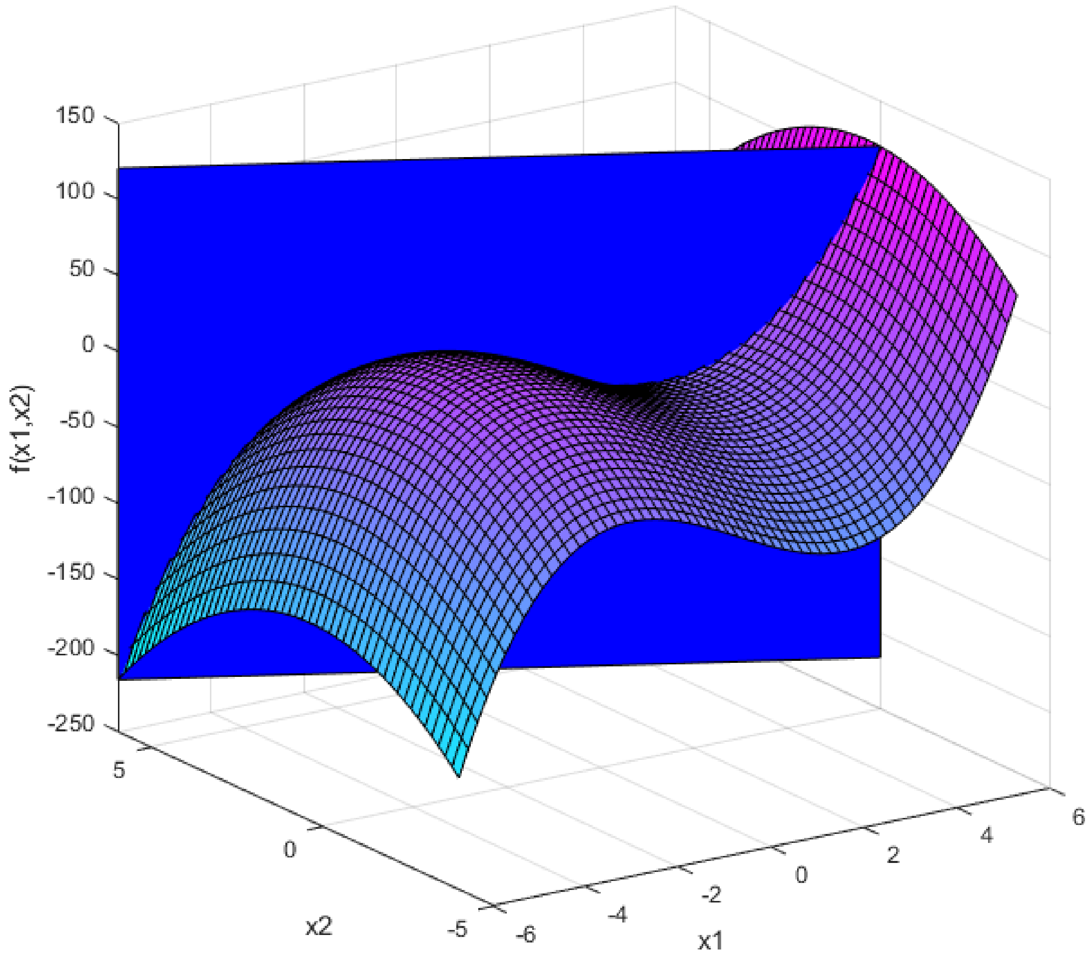

Figure Example 13: Plot of contour plot of the objective function and constraint. It shows that there is a local maximum and local minimum but with no global maximum of minimum values.

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Solution to Problem 14

Minimize & Maximize:	$f(x,y)=9x^2+13y^2+18xy-4$	(14)
Subject to:	$g_1(x,y)=x^2+y^2+2x\geq 16$	
Optimal Solution	$x^*_{\min_{Global}}=2.59, y^*_{\min_{Global}}=-2.01, f^*_{\min_{Global}}(2.59,-2.01)=15.25$ $x^*_{\min_{Local}}=-3.73, y^*_{\min_{Local}}=3.09, f^*_{\min_{Local}}(-3.73,3.09)=37.87$ No Local Maximum	

Inequality Constraints:

$$g_1(x,y)=x^2+y^2+2x\geq 16$$
$$\Rightarrow y^2\geq 16-x^2-2x$$
$$\therefore y\geq \pm\sqrt{16-x^2-2x}$$

Objective Function:

$$f(x,y)=9x^2+13y^2+18xy-4$$
$$\Rightarrow 9x^2+13y^2+18xy-4-f=0$$
$$\Rightarrow 13y^2+18xy+9x^2-4-f=0$$
$$\therefore 13y^2+(18x)y+\left(9x^2-4-f\right)=0$$

This rearranged objective function is in the form of a quadratic equation of the form:

$$ay^2+by+c=0$$

where:

$$a=13$$
$$b=18x$$
$$c=9x^2-4-f$$

The solution to this quadratic equation is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Which then gives that the solution of y is:

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-18x\pm\sqrt{(18x)^2-4\times 13\times\left(9x^2-4-f\right)}}{2\times 13}$$
$$\Rightarrow y=\frac{-18x\pm\sqrt{324x^2-468x^2+208+52f}}{26}$$
$$\therefore y=\frac{-18x\pm\sqrt{-144x^2+208+52f}}{26}$$

Calculating the constraints and objective function values for y between the limits of -4 and 4 for x

			fmin		fmin2	
			15.291		**37.877**	
x	**y(+g)**	**y(-g)**	**y(+f15.3)**	**y(-f15.3)**	**y(+f37.9)**	**y(-f37.9)**
-4.0	2.83	-2.83	#NUM!	#NUM!	#NUM!	#NUM!
-3.9	2.93	-2.93	#NUM!	#NUM!	#NUM!	#NUM!

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-3.8	3.03	-3.03	#NUM!	#NUM!	3.012	2.250
-3.7323	3.09	-3.09	#NUM!	#NUM!	**3.0878**	2.080
-3.6	3.20	-3.20	#NUM!	#NUM!	3.171	1.814
-3.5	3.28	-3.28	#NUM!	#NUM!	3.205	1.641
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.4	2.33	-2.33	-1.155	-2.168	-0.249	-3.074
2.5	2.18	-2.18	-1.340	-2.121	-0.356	-3.106
2.5945	2.02	**-2.0198**	-1.573	**-2.0198**	-0.459	-3.133
2.7	1.82	-1.82	#NUM!	#NUM!	-0.578	-3.161
2.8	1.60	-1.60	#NUM!	#NUM!	-0.693	-3.184
⋮	⋮	⋮	⋮	⋮	⋮	⋮
3.7	#NUM!	#NUM!	#NUM!	#NUM!	-2.009	-3.114
3.8	#NUM!	#NUM!	#NUM!	#NUM!	-2.250	-3.012
3.9	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
4.0	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!

Plot of optimal solutions

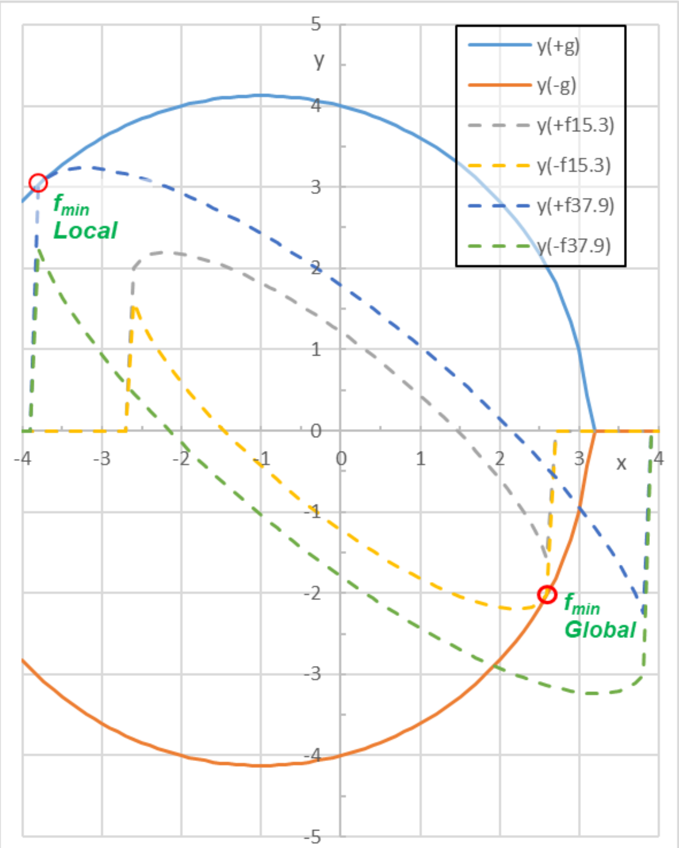

Figure Example 14: Plot of solution to problem 14, with the contours of the objective function showing the global and local minimum values and their corresponding contours as there is no local maximum

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2) Solution to Lecture Example Problems
Solution to Problem 15

Example 01: Design of a Can

$$\begin{aligned} &\textbf{Minimize:} && S(D,H) = \pi DH + \frac{\pi}{2}D^2 \\ &\textbf{Subject to:} && g_1(D,H) = 400 - \frac{\pi}{4}D^2H \le 0 \\ &\textbf{such that:} && 3.5 \le D \le 8 \\ & && 8 \le H \le 18 \end{aligned} \qquad (15)$$

Optimal Solution $\quad D^*_{\min} = 7.98, H^*_{\min} = 8, S^*_{\min_{Global}}(7.98,8) = 300.53$

1) *Solving this problem as a function of H*

Inequality Constraints:

$$g_1(D,H) = 400 - \frac{\pi}{4}D^2H \le 0$$

$$\Rightarrow 400 \le \frac{\pi}{4}D^2H$$

$$\Rightarrow D^2 \ge \frac{400}{H} \times \frac{4}{\pi} = \frac{1600}{\pi H}$$

$$\Rightarrow D \ge \pm\sqrt{\frac{1600}{\pi H}} = \pm\frac{40}{\sqrt{\pi H}}$$

$$\therefore D \ge \frac{40}{\sqrt{\pi H}}$$

Limits:

$$3.5 \le D \le 8 \qquad 8 \le H \le 18$$

Objective Function:

$$S(D,H) = \pi DH + \frac{\pi}{2}D^2$$

$$\Rightarrow \pi DH + \frac{\pi}{2}D^2 - S = 0$$

$$\Rightarrow \frac{\pi}{2}D^2 + (\pi H)D - S = 0$$

This rearranged objective function is in the form of a quadratic equation of the form:

$$ay^2 + by + c = 0$$

where:

$$a = \frac{\pi}{2} \qquad b = \pi H \qquad c = -S$$

The solution to this quadratic equation is:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Which then gives that the solution of y is:

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$$D=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\pi H\pm\sqrt{(\pi H)^2-4\times\frac{\pi}{2}\times(-S)}}{2\times\frac{\pi}{2}}$$

$$\Rightarrow D=\frac{-\pi H\pm\sqrt{\pi^2H^2+2\times\pi\times S}}{\pi}\ \therefore D=\frac{-\pi H\pm\sqrt{\pi^2H^2+2\pi S}}{\pi}$$

Calculating the constraints and objective function values for D between the limits of 8 and 18 for H.

		S = 300.53		S = 350		S = 400		S = 450	
H	**D(g1)**	**D(+S300)**	**D(-S300)**	**D(+S350)**	**D(-S350)**	**D(+S300)**	**D(-S400)**	**D(+S450)**	**D(-S450)**
8.0	**7.9788**	7.9788	-24.13	8.94	-24.94	9.85	-25.85	10.72	-26.72
8.1	7.929	7.929	-24.28	8.88	-25.08	9.80	-26.00	10.66	-26.86
8.2	7.881	7.880	-24.43	8.83	-25.23	9.74	-26.14	10.61	-27.01
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17.8	5.349	4.742	-40.34	5.43	-41.03	6.11	-41.71	6.76	-42.36
17.9	5.334	4.722	-40.52	5.41	-41.21	6.08	-41.88	6.74	-42.54
18.0	5.319	4.701	-40.70	5.38	-41.38	6.06	-42.06	6.71	-42.71

Plot of optimal solutions

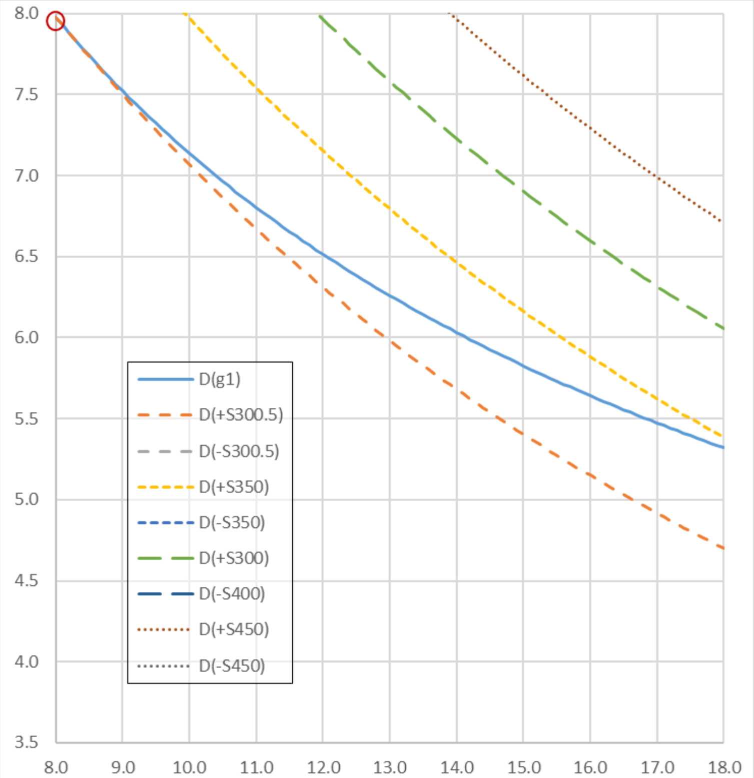

Figure Example 15: Plot of solution to problem 15 solving the problem as a function of H, with the contours of the objective function showing the global and local minimum values and their corresponding contours.

2) *Solving this problem as a function of D*

Inequality Constraints:

$$g_1(D,H)=400-\frac{\pi}{4}D^2H\le 0,\qquad \Rightarrow 400\le\frac{\pi}{4}D^2H$$

$$\Rightarrow H\ge\frac{400}{D^2}\times\frac{4}{\pi}=\frac{1600}{\pi D^2},\qquad \therefore H\ge\frac{1600}{\pi D^2}$$

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Limits:

$$3.5 \le D \le 8 \qquad 8 \le H \le 18$$

Objective Function:

$$S(D,H) = \pi DH + \frac{\pi}{2}D^2 \Rightarrow S - \frac{\pi}{2}D^2 = \pi DH \Rightarrow \pi DH = S - \frac{\pi}{2}D^2$$

$$\Rightarrow H = \frac{S - \frac{\pi}{2}D^2}{\pi D} = \frac{S}{\pi D} - \frac{\pi}{2}\frac{D^2}{\pi D} \qquad \therefore H = \frac{S}{\pi D} - \frac{D}{2}$$

Calculating the constraints and objective function values for H between the limits of 3.5 and 8 for D.

		S=	S=	S=	S=
		300.53	350.00	400.00	450.00
D	**H(g1)**	**D(S300.5)**	**D(S350)**	**D(S400)**	**D(S450)**
3.5	41.575	25.582	30.081	34.628	39.176
3.6	39.298	24.773	29.147	33.568	37.989
3.7	37.202	24.005	28.260	32.562	36.863
⋮	⋮	⋮	⋮	⋮	⋮
7.7	8.590	8.574	10.619	12.686	14.753
7.8	8.371	8.364	10.383	12.424	14.464
7.9	8.160	8.159	10.152	12.167	14.182
8.0	**7.958**	**7.958**	9.926	11.915	13.905

Plot of optimal solutions

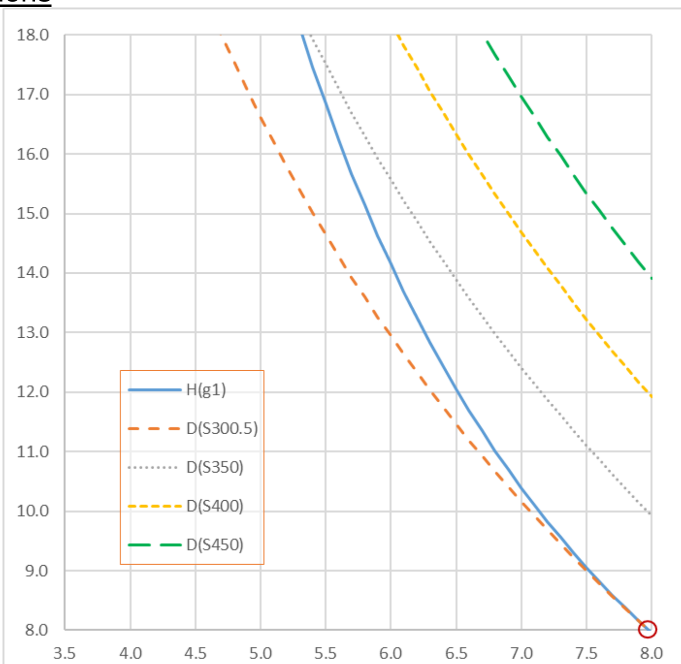

Figure Example 15: Plot of solution to problem 15 solving the problem as a function of D, with the contours of the objective function showing the global and local minimum values and their corresponding contours.

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Solution to Problem 16

Example 02: Design of a Rectangular Beam

$$\begin{aligned} \textbf{Minimize:} \quad & A(b,d) = bd \\ \textbf{Subject to:} \quad & g_1(b,d) = 840\times10^6 - 165bd^2 \le 0 \\ & g_2(b,d) = 36{,}000 - 50bd \le 0 \\ & g_3(b,d) = d - 2b \le 0 \\ \textbf{such that:} \quad & b,d \ge 0 \end{aligned} \tag{16}$$

Optimal Solution $b^*_{\min} = 108.371, d^*_{\min} = 216.741, S^*_{\min_{Global}}(108.371, 216.741) = 23488.41$

Inequality Constraints:

$g_1(b,d) = 840\times10^6 - 165bd^2 \le 0$

$\Rightarrow 840\times10^6 \le 165bd^2$

$\Rightarrow b \ge \dfrac{840\times10^6}{165d^2}$

$\therefore b \ge \dfrac{5.0909\times10^6}{d^2}$

$g_2(b,d) = 36{,}000 - 50bd \le 0$

$\Rightarrow 36{,}000 \le 50bd$

$\Rightarrow b \ge \dfrac{36{,}000}{50d}$

$\therefore b \ge \dfrac{720}{d}$

$g_3(b,d) = d - 2b \le 0$

$\Rightarrow d \le 2b$

$\therefore b \ge \dfrac{d}{2}$

Limits:

$b,d \ge 0$

Objective Function:

$A(b,d) = bd \quad \therefore b = \dfrac{A}{d}$

1) *Solving this problem only as a function of d*

Optimal Solution	
b =	108.371
d =	216.741
A(b,d) =	23488.41
g_1(b,d) =	0
g_2(b,d) =	-1138420
g_3(b,d) =	0

Calculating the constraints and objective function values for b between the limits of 100 and 400 for d.

				A=	A=	A=	A=	A=
				23488	50000	40000	30000	15000
d	**b(g1)**	**b(g2)**	**b(g3)**	**b(A23.6k)**	**b(A50k)**	**b(A40k)**	**b(A30k)**	**b(A15k)**
100	509.09	7.20	50.0	234.9	500.0	400.0	300.0	150.0
105	461.76	6.86	52.5	223.7	476.2	381.0	285.7	142.9

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110	420.74	6.55	55.0	213.5	454.5	363.6	272.7	136.4
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
200	127.27	3.60	100.0	117.4	250.0	200.0	150.0	75.0
205	121.14	3.51	102.5	114.6	243.9	195.1	146.3	73.2
210	115.44	3.43	105.0	111.8	238.1	190.5	142.9	71.4
216.74	**108.37**	3.32	108.4	**108.37**	230.7	184.6	138.4	69.2
220	105.18	3.27	110.0	106.8	227.3	181.8	136.4	68.2
225	100.56	3.20	112.5	104.4	222.2	177.8	133.3	66.7
230	96.24	3.13	115.0	102.1	217.4	173.9	130.4	65.2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
390	33.47	1.85	195.0	60.2	128.2	102.6	76.9	38.5
395	32.63	1.82	197.5	59.5	126.6	101.3	75.9	38.0
400	31.82	1.80	200.0	58.7	125.0	100.0	75.0	37.5

Plot of optimal solutions

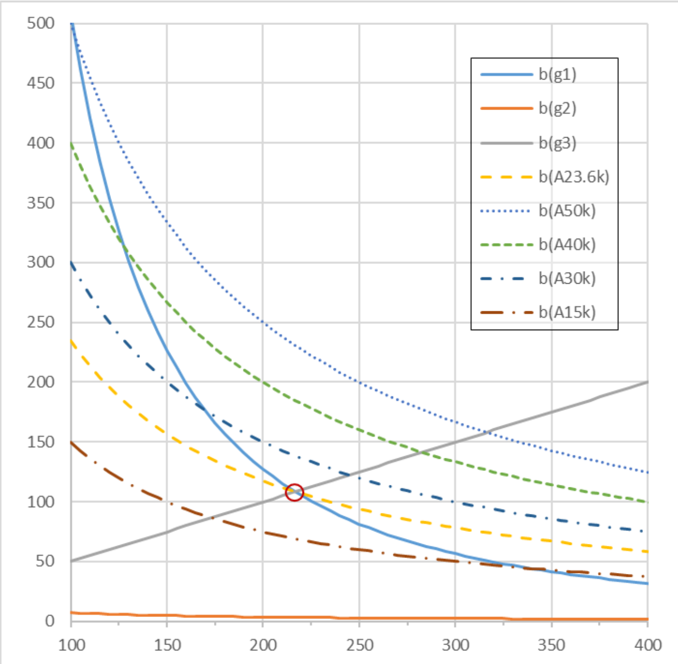

Figure Example 16: Plot of solution to problem 16 solving the problem as a function of d, with the contours of the objective function showing the global and local minimum values and their corresponding contours.

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